V. I. Gurarii

1962

SovietRxiv

Abstract

Full Text

V. I. Gurarii

ON INCLINATIONS OF SUBSPACES AND CONDITIONAL BASES IN BANACH SPACE

(Presented by Academician A. N. Kolmogorov, 23 II 1962)

Let P and Q be linear manifolds in a Banach space E. Define the inclination of P to Q as the quantity

$$(\widehat{P;\,Q}) = \inf_{\substack{x\in P,\ y\in Q\\ \|x\|=1}} \|x+y\|.$$

Denote by $L_{\{f_1,\dots,f_n\}}$ the linear span of the elements $f_1,\dots,f_n$; $f_i \in E$ $(i = 1,2,\dots,n)$, and call the index of the collection $\{f_i\}_{i=1}^n$ the quantity

$$\gamma_{\{f_1,\dots,f_n\}} = \min_{1\le p<n} \left(L_{\{f_1,\dots,f_p\}};\hat{}\, L_{\{f_{p+1},\dots,f_n\}}\right).$$

Analogously, the index of a sequence $\{f_i\}_{i=1}^\infty$, $f_i \in E$, is defined as

$$\gamma_{\{f_1,\dots,f_n,\dots\}} = \inf_n \gamma_{\{f_1,\dots,f_n\}}.$$

As was shown by M. M. Grinblum ([1]), in order that a sequence $\{f_i\}_{i=1}^\infty$ complete in E be a basis* in E, it is necessary and sufficient that the condition $\gamma_{\{f_1,f_2,\dots\}} > 0$ be satisfied.

From the results of K. I. Babenko ([2]) and V. G. Vinokurov ([3]) it follows that in Hilbert spaces and in some classes of Banach spaces there exist subspaces with conditional bases. Here a generalization of this assertion to arbitrary infinite-dimensional Banach spaces will be obtained. Theorems 1–4 will serve as the apparatus for obtaining the results.

Theorem 1. *For given $\varepsilon > 0$ and a finite-dimensional subspace $P \subset E$ of an infinite-dimensional Banach space E, there exists an infinite-dimensional subspace $Q \subset E$ such that $(\widehat{P;\,Q}) > 1-\varepsilon$.*

Proof. On the basis of the Banach–Mazur theorem one may regard E as a subspace of $C_{[0;1]}$. The unit sphere S_P of the finite-dimensional subspace P is compact in $C_{[0;1]}$ and, consequently, constitutes a family of functions equicontinuous on $[0;1]$. Therefore, for the given $\varepsilon > 0$ there is a natural number n such that if $|x_1 - x_2| < 1/n$, then $|f(x_1) - f(x_2)| < \varepsilon$ for every $f(x) \in S_P$.

Consider the set Q of functions from E that vanish at the points $x_k = k/n,\ k = 0, 1, \dots, n$. It is not difficult to verify that Q is an infinite-dimensional subspace of the space E.

---

* A sequence $\{f_i\}_1^\infty$ is called a basis in E if every element $f \in E$ can be represented in a unique way in the form

$$f = \sum_{i=1}^{\infty} \alpha_i f_i.$$

A sequence $\{f_i\}_1^\infty$ that remains a basis in E under any permutation of its elements is called an unconditional basis in E. A basis that is not unconditional is called a conditional basis.

For the given $f(x) \in S_P$ and $g(x) \in Q$, let $\tilde{x}$ be one of the points at which $|f(x)|$ assumes its maximum; then for some natural $i_0 \leqslant n$,

$$\frac{i_0 - 1}{n} \leqslant \tilde{x} \leqslant \frac{i_0}{n}.$$

We have:

$$\max_{x\in[0;1]} |f(x) - g(x)| \geqslant \max_{i=0,1,\dots,n} \left| f\left(\frac{i}{n}\right) - g\left(\frac{i}{n}\right) \right| =$$

$$= \max_{i=0,1,\dots,n} \left| f\left(\frac{i}{n}\right) \right| \geqslant \left| f\left(\frac{i_0}{n}\right) \right| \geqslant |f(\tilde{x})| - \varepsilon = 1 - \varepsilon.$$

This inequality means that $(\widehat{P;Q}) \geqslant 1 - \varepsilon$. The theorem is proved.

Lemma. Let, for linear manifolds $P_1, P_2 = Q + R$, and P_3 in a Banach space, the following conditions be satisfied:

1. $(\widehat{Q;R}) \geqslant \alpha > 0$.
2. $(P_1 + \widehat{P_2; P_3}) \geqslant \beta; \quad (P_1; \widehat{P_2 + P_3}) \geqslant \beta; \quad \beta > 0.$

Then

$$(P_1 + Q; R + P_3) \geqslant \frac{\alpha\beta}{2 + \alpha}.$$

From the lemma there follows directly the following theorem on the possibility of "synthesizing" a basis in a space from bases in its subspaces.

Theorem 2. Let $\{f_i^{(1)}, f_i^{(2)}, \dots, f_i^{(k_i)}\}$, $i = 1, 2, \dots$, be bases in k_i-dimensional subspaces $P_i \subset E$, and let

$$\gamma_{\{f_i^{(1)}, \dots, f_i^{(k_i)}\}} \geqslant \alpha > 0, \quad i = 1, 2, \dots,$$

and, for any natural m, n, $m < n$,

$$(P_1 + \dots + \widehat{P_m}; P_{m+1} + \dots + P_n) \geqslant \beta > 0.$$

Then the sequence

$$f_1^{(1)}, \dots, f_1^{(k_1)}, f_2^{(1)}, \dots, f_2^{(k_2)}, \dots, f_i^{(1)}, \dots, f_i^{(k_i)}, \dots$$

is a basis in its closed linear span, and

$$\gamma_{\{f_1^{(1)}, \dots, f_1^{(k_1)}, \dots, f_2^{(1)}, \dots\}} \geqslant \frac{\alpha\beta}{2 + \alpha}.$$

The result obtained below makes essential use of an important theorem on ε-isometry of Banach spaces due to A. Dvoretzky ([4]).

Definition. Banach spaces E_1 and E_2 are called ε-isometric if there exists an isomorphic mapping T from E_1 onto E_2 such that, for any $f \in E_1$,

$$(1 - \varepsilon)\|f\| \leqslant \|Tf\| \leqslant (1 + \varepsilon)\|f\|.$$

Theorem 3 (A. Dvoretzky). For every $\varepsilon > 0$ and natural n there exists a natural $N_0(n; \varepsilon)$ such that every N-dimensional Banach space, for $N > N_0$, contains an n-dimensional subspace ε-isometric to n-dimensional Euclidean space.

From the cited results of K. I. Babenko, M. M. Grinblyum, and A. Dvoretzky it follows:

Theorem 4. In an infinite-dimensional Banach space E, for every $\varepsilon > 0$ there can be found a collection of elements $f_1, \dots, f_n$, $n = n(\varepsilon)$, such that $\gamma_{\{f_1, \dots, f_n\}} > \alpha > 0$, where α is an absolute constant, and under a suitable permutation of the elements the index of this collection becomes less than ε.

Theorem 5. In every infinite-dimensional Banach space there exists a sequence $\{f_i\}_{i=1}^{\infty}$ which is a conditional basis in the closure of its linear span.

Proof. Consider a sequence of numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ satisfying the conditions: $0 < \varepsilon_i < 1$ and

$$\prod_{i=1}^{\infty}(1 - \varepsilon_i) = \beta > 0.$$

By Theorem 4, in the given Banach space E there exists a collection of elements $\{f_1^{(1)}, f_1^{(2)}, \dots, f_1^{(k_1)}\}$ such that

$$\gamma_{\{f_1^{(1)}, \dots, f_1^{(k_1)}\}} > \alpha > 0,$$

but, under a suitable permutation of its elements, we shall have

$$\gamma_{\{\tilde{f}_1^{(1)},\dots,\tilde{f}_1^{(k_1)}\}} < \varepsilon_1.$$

Denote by P_1 the subspace spanned by $f_1^{(1)}, \dots, f_1^{(k_1)}$, and consider an infinite-dimensional subspace $Q_1 \subset E$ such that

$$(P_1, Q_1) > 1 - \varepsilon_1$$

(the existence of such a subspace is ensured by Theorem 1).

In the subspace Q_1 there exists a collection of elements $\{f_2^{(1)}, \dots, f_2^{(k_2)}\}$ such that

$$\gamma_{\{f_2^{(1)},\dots,f_2^{(k_2)}\}} > \alpha,$$

but, under a suitable permutation of the elements, we shall have

$$\gamma_{\{\tilde{f}_2^{(1)},\dots,\tilde{f}_2^{(k_2)}\}} < \varepsilon_2.$$

Denote by P_2 the subspace spanned by $f_2^{(1)}, \dots, f_2^{(k_2)}$, and consider an infinite-dimensional subspace $Q_2 \subset E$ such that

$$(P_1 + P_2, Q_2) > 1 - \varepsilon_2.$$

Continuing this process indefinitely, we obtain a collection of a sequence of finite-dimensional subspaces $\{P_i\}_{i=1}^{\infty}$ with bases $\{f_i^{(1)}, \dots, f_i^{(k_i)}\}$, for which we have

$$\gamma_{\{f_i^{(1)},\dots,f_i^{(k_i)}\}} > \alpha$$

and

$$\gamma_{\{\tilde{f}_i^{(1)},\dots,\tilde{f}_i^{(k_i)}\}} < \varepsilon_i$$

(the collection $\{\tilde{f}_i^{(1)}, \dots, \tilde{f}_i^{(k_i)}\}$ is obtained from the collection $\{f_i^{(1)}, \dots, f_i^{(k_i)}\}$ by some permutation of the elements), and, moreover,

$$(P_1 + P_2 + \dots + P_i, Q_i) > 1 - \varepsilon_i, \qquad i = 1, 2, \dots .$$

Since $P_{i+1} \subset Q_i$, it follows that

$$(P_1 + P_2 + \dots + P_i, P_{i+1}) \geq (P_1 + P_2 + \dots + P_i, Q_i) > 1 - \varepsilon_i.$$

Let us show that the sequence

$$\{f_1^{(1)}, \dots, f_1^{(k_1)}, f_2^{(1)}, \dots, f_2^{(k_2)}, \dots\}$$

forms a basis in the subspace $P \subset E$ spanned by it.

Let

$$f \in P_1 + \dots + P_m, \qquad g \in P_{m+1} + \dots + P_{m+n}, \qquad g = g_1 + g_2 + \dots + g_n,$$

where $g_k \in P_{m+k}$, $k = 1, 2, \dots, n$. We have:

$$\begin{aligned}\|f + g\| &= \|(f + g_1 + \dots + g_{n-1}) + g_n\| \\ &\geq (1 - \varepsilon_{m+n-1})\|f + g_1 + \dots + g_{n-1}\| \\ &\geq (1 - \varepsilon_{m+n-1})(1 - \varepsilon_{m+n-2})\|f + g_1 + \dots + g_{n-2}\| \geq \cdots \\ &\geq (1 - \varepsilon_{m+n-1})(1 - \varepsilon_{m+n-2}) \cdots (1 - \varepsilon_m)\|f\| \geq \beta\|f\|.\end{aligned}$$

This inequality means that

$$(P_1 + \dots + P_m,\, P_{m+1} + \dots + P_{m+n}) \geq \beta,$$

and, by Theorem 2, the sequence

$$f_1^{(1)}, \dots, f_1^{(k_1)}, f_2^{(1)}, \dots, f_2^{(k_2)}, \dots$$

is a basis in P. This basis is conditional, since by a permutation of its elements one can obtain the sequence

$$\tilde{f}_1^{(1)}, \dots, \tilde{f}_1^{(k_1)}, \tilde{f}_2^{(1)}, \dots, \tilde{f}_2^{(k_2)}, \dots,$$

for which, in view of the fact that $\lim_{i\to\infty} \varepsilon_i = 0$, we have:

$$\gamma_{\{\tilde{f}_1^{(1)}, \dots, \tilde{f}_1^{(k_1)}, \tilde{f}_2^{(1)}, \dots, \tilde{f}_2^{(k_2)}, \dots\}} = 0.$$

The theorem is proved.

Kharkov Automobile and Highway Institute

Received
23 II 1962

References

1. M. M. Grinblyum, DAN, **31**, 428 (1941).

2. K. I. Babenko, DAN, **62**, 157 (1948).

3. V. G. Vinokurov, DAN, **85**, 685 (1952).

4. A. Dvoretzky, Proc. Nat. Acad. Sci. USA, **45**, 223 (1959).

Note: Figure translations are in progress. See original paper for figures.