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1962

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Abstract

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MATHEMATICS

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ON A BOUNDARY ESTIMATE FOR THE SOLUTION OF AN ELLIPTIC-TYPE EQUATION OF ARBITRARY ORDER WITH CONSTANT COEFFICIENTS

(Presented by Academician N. N. Bogolyubov, 31 III 1962)

§ 1. The estimates available in the literature (^{1}) for the growth of solutions of an elliptic-type equation near the boundary of their domain of definition were obtained in the metric C, chiefly on the basis of a study of the corresponding Green' s function. S. M. Nikol' skii (^{2}) obtained, by another method, estimates exact in order for the growth of a harmonic function and its derivatives near the boundary of a domain in the sense of L_p.

The present work is devoted to obtaining similar estimates in the metric L_2, but for a general elliptic-type equation with constant coefficients of arbitrary order in an arbitrary bounded N-dimensional domain with sufficiently smooth boundary. The investigations are in fact carried out in the two-dimensional case. They are transferred to the N-dimensional case by analogy.

In the case of two dimensions we are concerned with a solution u of a differential equation that is the Euler equation for the functional (Dirichlet integral):

$$D_G^{(l)}(u) = \iint_G \left[\sum_{i=0}^{l} a_i \left(\frac{\partial^l u}{\partial x^{l-i} \partial y^i} \right)^2 + 2 \sum_{\substack{i,p=0 \\ i \neq p}}^{l} b_{ip} \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \frac{\partial^l u}{\partial x^{l-p} \partial y^p} \right] dx\, dy. \tag{1,1}$$

It is assumed that G is a bounded domain with boundary Γ of class $C^{(l+1)}$, and that the coefficients a_i, b_{ip} satisfy the inequality

$$k \sum_{i=0}^{l} \xi_i^2 \leq \sum_{i=0}^{l} a_i \xi_i^2 + 2 \sum_{\substack{i,p=0 \\ i \neq p}}^{l} b_{ip} \xi_i \xi_p, \tag{1,2}$$

where $k > 0,\ b_{ip} = b_{pi}$.

§ 2. Let $0 < h < \delta/2 < \delta < 1$, $l > 0$ be a natural number, and

$$\lambda_m(t) = \begin{cases} C_m(h) \displaystyle\int_h^t (t-h)^{2l+2m-1}(\delta - t)^l \, dt, & (h \le t \le \delta), \\ 0, & (t \le h), \\ 1, & (\delta \le t;\ 0 \le t \le +\infty), \end{cases} \tag{2,1}$$

where $C_m(h)$ is determined from the condition $\lambda_m(\delta) = 1$.

The function $\lambda_m(t) \ge 0$ is continuous together with its derivatives up to order l inclusive on the whole real semiaxis and satisfies on $[h, \delta]$ the inequalities:

$$\left(|\lambda_m^{(s)}(t)|\right)^{\frac{2(l+m)}{2l+2m-s}} \le A_m \lambda_m(t); \tag{2,2}$$

$$\left(|\lambda_m^{(s)}(t)|\right)^{\frac{2(l+m-s)}{2l+2m-s}} \le B_m (t-h)^{2(l-s+m)}; \tag{2,3}$$

$$(t-h)^{2(l+m)} \le D_m \lambda_m(t), \tag{2,4}$$

where A_m, B_m, D_m are constants independent of h and t; $m > 0$ is arbitrary; $s < l + m$ is an integer.

§ 3. Introduce, near the boundary Γ, a new coordinate system (s, t): t is the distance of the point to Γ in the direction of the normal, and s is the length of the arc of Γ. Let G_δ be the domain obtained by removing from G the strip determined by the inequality $0 \leqslant t \leqslant \delta$.

We construct the auxiliary function $\eta(x, y)$ in the following way:

$$\eta(x,y) = \begin{cases} 0 & \text{on } G - G_h, \\ \lambda_m(t) & \text{on } G_h - G_\delta = \Pi_{h,\delta}, \\ 1 & \text{on } G_\delta. \end{cases} \tag{3,1}$$

Since $\Gamma \in C^{(l+1)}$, the partial derivatives with respect to x and y of $\eta(x, y)$ will be expressed linearly through the partial derivatives with respect to the variables s and t introduced near the boundary, and the inequality

$$\left| \frac{\partial^p \eta}{\partial x^{p-i} \, \partial y^i} \right| \leqslant \sum_{k=1}^{p} |\psi_k(s,t)| \cdot |\lambda_m^{(k)}(t)| \leqslant M |\lambda_m^{(p)}(t)|, \tag{3,2}$$

will hold, where $\psi_k(s, t)$ are continuous (bounded) functions on $\Pi_{h,\delta}$,

$$M = p \max_{(k)} |\psi_k(s,t)|, \qquad p = 1, 2, \dots, l.$$

§ 4. Let a function $u(x,y)$ be defined on G, having generalized derivatives up to order l inclusive and such that, for every $h > 0$ (see (1,1)),

$$D^{(l)}_{G_h}(u) < +\infty. \tag{4,1}$$

It follows from (4,1) and (1,2) that the function $u(x,y)$ has generalized derivatives of order l with integrable square on G_h.

We impose on the function u the following additional conditions, namely: whatever $h > 0$ may be, and for any function $v \subset D^{(l)}_{G_h}(v) < +\infty$ such that

$$\left.\frac{\partial^k v}{\partial n^k}\right|_{\Gamma_h} = 0 \qquad (k = 0, 1, \dots, l-1) \tag{4,2}$$

(where Γ_h is the boundary of the domain G_h, and n is the inward normal), the equality holds:

$$D^{(l)}_{G_h}(u,v) = \iint_{G_h} \left[\sum_{i=0}^{l} a_i \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \frac{\partial^l v}{\partial x^{l-i} \partial y^i} + \right.$$

$$\left. + \sum_{\substack{i,p=0 \\ i \neq p}}^{l} b_{ip} \left(\frac{\partial^l u}{\partial x^{l-i} \partial y^i} \frac{\partial^l v}{\partial x^{l-p} \partial y^p} + \frac{\partial^l u}{\partial x^{l-p} \partial y^p} \frac{\partial^l v}{\partial x^{l-i} \partial y^i} \right) \right] dx\, dy = 0. \tag{4,3}$$

Then u is called a generalized solution of the Euler equation corresponding to the functional (1,1).

The equality holds:

$$D^{(l)}_{G_h}(u, u\eta) = \iint_{G_h} \left\{ \sum_{i=0}^{l} a_i \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \sum_{s=0}^{l} \sum_{j=0}^{i} \frac{\partial^s \eta}{\partial y^j \partial x^{s-j}} \frac{\partial^{l-s} u}{\partial x^{l-i-s+j} \partial y^{i-j}} C_l^{s-j} C_i^j + \right.$$

$$+ \sum_{\substack{i,p=0 \\ i \neq p}}^{l} b_{ip} \left[\left(\sum_{s=0}^{l} \sum_{j=0}^{i} \frac{\partial^s \eta}{\partial y^j \partial x^{s-j}} \frac{\partial^{l-s} u}{\partial x^{l-i-s+j} \partial y^{i-j}} C_{l-i}^{s-j} C_i^j \right) \frac{\partial^l u}{\partial x^{l-p} \partial y^p} + \right.$$

$$+ \left(\sum_{s=0}^{l} \sum_{j=0}^{p} \frac{\partial^s \eta}{\partial y^j \partial x^{s-j}} \frac{\partial^{l-s} u}{\partial x^{l-p-s+j} \partial y^{p-j}} C_{l-p}^{s-j} C_p^j \right) \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \Bigg] \Bigg\} \, dx \, dy = 0. \quad (4,4)$$

We single out the terms obtained for $s = 0$ on the left-hand side and estimate them with the aid of the remaining terms.

After applying Schwarz' s inequality, (3.2), (2.2), (2.3), and (1.2), we shall have:

$$\iint_{G_h} \left[\sum_{k=0}^{l} C_l^k \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 \right] \eta \, dx \, dy \ll$$

$$\ll C \left\{ \left[\iint_{\Pi_{h,\delta}} \sum_{k=0}^{l} C_l^k \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 (t-h)^{2l+2m} \, dx \, dy \right] \right\}^{1/2} \times$$

$$\times \sum_{s=1}^{l} \left\{ \left[\iint_{\Pi_{h,\delta}} \sum_{i=0}^{l-s} C_{l-s}^i \left(\frac{\partial^{l-s} u}{\partial x^{l-s-i} \partial y^i} \right)^2 (t-h)^{2(l+m-s)} \, dx \, dy \right] \right\}^{1/2}. \quad (4.5)$$

Introduce the notation:

$$I_{h,m}^{(l-s)} = \left[\iint_{\Pi_{h,\delta}} \sum_{k=0}^{l-s} C_{l-s}^k \left(\frac{\partial^{l-s} u}{\partial x^{l-s-k} \partial y^k} \right)^2 (t-h)^{2(l+m-s)} \, dx \, dy \right]^{1/2};$$

$$I_h = \left[\iint_{\Pi_{h,\delta}} \sum_{k=0}^{l} C_l^k \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 \eta \, dx \, dy \right]^{1/2};$$

$$I_{h,m}^{(0)} = \left[\iint_{\Pi_{h,\delta}} \sum_{k=0}^{l} C_l^k \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 (t-h)^{2l+2m} \, dx \, dy \right]^{1/2}; \quad (4.6)$$

$$I_{h,0}^{(0)} = I_h^{(0)}; \qquad \overline{I}_{h,m}^{(0)} = \left[\iint_{G_h} \sum_{k=0}^{l} \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 (t-h)^{2l+2m} \, dx \, dy \right]^{1/2};$$

$$I_\delta^{(0)} = \left[\iint_{G_\delta} \sum_{k=0}^{l} C_l^k \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 dx \, dy \right]^{1/2}.$$

From (4.5) and (4.6), taking (2.4) into account, we obtain the inequality

$$\overline{I}^{(0)}_{h,m} \ll \sqrt{A_{mB_{mD}} m}\, E \sum_{s=1}^{l} I^{(l-s)}_{h,m} \qquad (E \text{ is a constant}). \tag{4.7}$$

Next, letting h tend to zero, we obtain

$$\overline{I}^{(0)}_{0,m} \ll \sqrt{A_{mB_{mD}} m}\, E \sum_{s=1}^{l} I^{(l-s)}_{0,m}, \tag{4.7'}$$

and then, for $m = 0$, we have

$$\|u\|_{W^{(l)}_{2,-l}(G)} \ll C_1 \|u\|_{W^{(l-1)}_{2,-(l-1)}(\Pi_{0,\delta})}, \tag{4.8}$$

where, as usual, it is denoted that

$$\|u\|_{W^{(r)}_{2,-r}(G)} = \iint_G \sum_{k=0}^{r} \left(\frac{\partial^r u}{\partial x^{r-k} \partial y^k} \right)^2 t^{2r}\, dx\, dy. \tag{4.9}$$

Let us note that inequality (4.8) is also valid for variable coefficients $a_i(x, y)$, $b_{ip}(x, y)$, which satisfy condition (1.2); and the proof given above changes in no way in this case.

§ 5. If, instead of the function u, one considers its partial derivatives $\partial^r u / \partial x^{r-i} \partial y^i$ (where $r > 0$ is any integer, $i = 0, 1, 2, \dots, r$), then, in the case of constant a_i, b_{ip}, these partial derivatives will also be a solution of the Euler equation corresponding to the functional (1,1), and for them the equality (4,3) will hold. Therefore, for each of the indicated partial derivatives the inequality (4,7′) is valid. Applying it successively for $r = 1, 2, \dots$ and each time putting, respectively, $m = 1, 2, \dots$, we finally obtain

$$\|u\|_{W^{(r)}_{2,-r}(G)} \leq C_r \|u\|_{W^{(l-1)}_{2,-(l-1)}(\Pi_{0,\delta})}, \tag{5,1}$$

where $r = l, l + 1, \dots$; C_r is a constant; $\Pi_{0,\delta} = G - G_\delta$.

§ 6. The estimates obtained are sharp with respect to the order t, as is shown by the following example.

Consider, in the unit disk, the harmonic function ($l = 1$)

$$u = \sum_{n=N}^{\infty} a_n \rho^n \cos n\theta, \tag{6,1}$$

where

$$a_n^2 = \frac{1}{(n+1)^{\frac{2-\alpha}{2}}} \quad (1 < \alpha < 2). \tag{6,2}$$

For it,

$$I = \int_0^1 \int_0^{2\pi} u^2 \rho \, d\rho \, d\theta = \pi \sum_{n=N}^{\infty} \frac{1}{2(n+1)^{2+\frac{2-\alpha}{2}}} < +\infty. \tag{6,3}$$

On the other hand, the inequality holds:

$$\begin{aligned} I_1 &= \int_{1-\delta}^1 \int_0^{2\pi} \left(\frac{\partial u}{\partial \rho}\right)^2 (1-\rho)^\alpha \rho \, d\rho \, d\theta \\ &= \pi \sum_{n=N}^{\infty} n^2 a_n^2 \int_{1-\delta}^1 (1-\rho)^\alpha \rho^{2n+1} \, d\rho \geq \\ &\geq C(\alpha) \sum_{n=N}^{\infty} a_n^2 \frac{1}{(n+1)^{\alpha-1}} = C(\alpha) \sum_{n=N}^{\infty} \frac{1}{(n+1)^{\alpha/2}} = +\infty, \end{aligned} \tag{6,4}$$

where

$$N = N(\delta), \qquad C(\alpha) = \frac{3\pi}{2^{\alpha+3} e^{4\alpha}}.$$

§ 7. For arbitrary functions that are not solutions of equations of elliptic type, the inequalities (5,1) and (4,8), of course, do not hold. For example, for the function $\psi(t) = \dfrac{1}{\sqrt{t}\ln t}$

$$\int_0^\delta \psi^2 \, dt < +\infty, \qquad \text{but} \qquad \int_0^\delta \psi i'^2 t^2 \, dt = +\infty.$$

Consequently, for it (5,1) does not hold when $l = 1, \ r = 1$.

Received
22 III 1962

CITED LITERATURE

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2. S. M. Nikol' skii, *Siberian Mathematical Journal*, 1, No. 1 (1960).