# ON THE EMBEDDING OF SUBGROUPS

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## Abstract

## Full Text

## MATHEMATICS

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# ON THE EMBEDDING OF SUBGROUPS

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§ 1. In the present note a number of theorems obtained by us on the embedding of subgroups are given; as special cases, Theorems 2, 3, 5 of our paper ([3]) follow from them. Only finite groups are considered. We use the notation and definitions of our previous papers ([2,3]), as well as the following definition.

**Definition.** Let $\sigma$ and $\tau$ be such (empty or not) sets of prime numbers that $\sigma \cap \tau$ is empty. Then we shall say that the strong $N_\sigma^\tau$-Sylow theorem holds for a group $\mathfrak{G}$, if for the normalizer of every $\tau$-subgroup in the group $\mathfrak{G}$ the strong $\sigma$-Sylow theorem holds.

Since the identity subgroup $\mathfrak{E}$ of the group $\mathfrak{G}$ is a $\tau$-subgroup, it follows from the definition that the strong $\sigma$-Sylow theorem also holds for the group $\mathfrak{G}$.

§ 2. Let $\mathfrak{G}$ possess a subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{G}_\tau$, where $\mathfrak{H}_\sigma$ is a $\sigma$-Hall subgroup of $\mathfrak{H}$, and $\mathfrak{G}_\tau$ is some $\tau$-Hall subgroup of $\mathfrak{G}$. Then the following theorems hold:

**Theorem 1.** *If the strong $N_\sigma^\tau$-Sylow theorem and the $\tau$-Sylow theorem hold for $\mathfrak{G}$, then for any $\sigma$-solvable subgroup $\mathfrak{M}$ whose order divides $(\mathfrak{H})$, there exists an element $G \in \mathfrak{G}$ such that $\mathfrak{M}^G \subseteq \mathfrak{H}$.*

**Proof.** Suppose that the theorem is false for the group $\mathfrak{G}$. Then, among all $\sigma$-solvable subgroups of the group $\mathfrak{G}$ for which the assertion of the theorem does not hold, choose a subgroup $\mathfrak{M}$ of least order. $(\mathfrak{M}) = (\mathfrak{M})_\sigma(\mathfrak{M})_\tau$. Obviously, $(\mathfrak{M}) > 1$.

Consider the following possibilities:

1. $\mathfrak{M}$ contains an invariant subgroup $\mathfrak{M}_\tau$ of order $(\mathfrak{M})_\tau$. On the basis of Schur' s theorem, $\mathfrak{M}$ contains a subgroup $\mathfrak{M}_\sigma$ of order $(\mathfrak{M})_\sigma$. According to the hypothesis of the theorem, $\mathfrak{M}_\tau^* = \mathfrak{M}_\tau^{G_1} \subseteq \mathfrak{G}_\tau$, where $G_1 \in \mathfrak{G}$. Since $\mathfrak{M}_\tau^*$ is invariant in $\mathfrak{M}^* = \mathfrak{M}_\sigma^*\mathfrak{M}_\tau^*$, where $\mathfrak{M}_\sigma^* = \mathfrak{M}_\sigma^{G_1}$, the normalizer $\mathfrak{N}_{\mathfrak{M}_\tau^*}^{\mathfrak{G}}$ contains $\mathfrak{M}_\sigma^*$. It is also clear that $\mathfrak{N}_{\mathfrak{M}_\tau^*}^{\mathfrak{G}}$ contains $\mathfrak{H}_\sigma$. On the basis of the hypothesis of the theorem we can assert that $\mathfrak{M}_\sigma^{*N} \subseteq \mathfrak{H}_\sigma$, where $N \in \mathfrak{N}_{\mathfrak{M}_\tau^*}^{\mathfrak{G}}$. Then

$$\mathfrak{M}^{*N} = \mathfrak{M}_\sigma^{*N}\mathfrak{M}_\tau^{*N} = \mathfrak{M}_\sigma^{*N}\mathfrak{M}_\tau^* \subseteq \mathfrak{H}_\sigma \times \mathfrak{G}_\tau = \mathfrak{H}.$$

Consequently, $\mathfrak{M}^G \subseteq \mathfrak{H}$, where $G = G_1N$. Contradiction.

2. $\mathfrak{M}$ is any $\sigma$-solvable subgroup of the group $\mathfrak{G}$. Then each index of a certain composition series
$$\mathfrak{M} \supset \mathfrak{M}^{(1)} \supset \cdots \supset \mathfrak{E}$$
of the group $\mathfrak{M}$ either is not divisible by any prime number from $\sigma$, or is some prime number belonging to $\sigma$.

Consider the following cases:

a) The index of $\mathfrak{M}^{(1)}$ in $\mathfrak{M}$ is not divisible by any prime number from $\sigma$, i.e.
$$(\mathfrak{M}^{(1)}) = (\mathfrak{M})_\sigma (\mathfrak{M}^{(1)})_\tau.$$
Since $\mathfrak{M}^{(1)}$ is $\sigma$-solvable and $(\mathfrak{M}^{(1)}) < (\mathfrak{M})$, it follows that
$$\mathfrak{M}^{(1)G_1} \subseteq \mathfrak{H},$$
where $G_1 \in \mathfrak{G}$. Consequently, $\mathfrak{M}^{(1)}$ contains an invariant solvable sub-

the group $\mathfrak{M}_\sigma^{(1)} = \mathfrak{M}_\sigma$, which, as is easy to see, will also be invariant in $\mathfrak{M}$. By Schur' s theorem, in $\mathfrak{M}$ there exists a subgroup $\mathfrak{M}_\tau$ of order $(\mathfrak{M})_\tau$.

Suppose first that $(\mathfrak{M})_\tau$ is divisible by only one prime number from $\tau$. Since the strong $\sigma$-Sylow theorem holds for the group $\mathfrak{G}$, there exists an element $G_2$ in the group $\mathfrak{G}$ such that
$$\mathfrak{M}_\sigma^* = \mathfrak{M}_\sigma^{G_2} \subseteq \mathfrak{H}_\sigma.$$
It is clear that the normalizer $N_{\mathfrak{G}}^{\mathfrak{M}_\sigma{}^*}$ contains $\mathfrak{M}_\tau^* = \mathfrak{M}_\tau^{G_2}$ and $\mathfrak{G}_\sigma$. By Sylow' s theorem,
$$\mathfrak{M}_\tau^{*N} \subseteq \mathfrak{G}_\tau,$$
where
$$N \in N_{\mathfrak{G}}^{\mathfrak{M}_\sigma{}^*}.$$
Consequently,
$$\mathfrak{M}^{*N} = \mathfrak{M}_\sigma^{*N} \mathfrak{M}_\tau^{*N} = \mathfrak{M}_\sigma^* \mathfrak{M}_\sigma^{*N} \subseteq \mathfrak{H}_\sigma \times \mathfrak{G}_\tau = \mathfrak{H},$$
or $\mathfrak{M}^G \subseteq \mathfrak{H}$, where $G = G_2 N$. We have a contradiction. Now suppose that $(\mathfrak{M})_\tau$ is divisible by more than one prime number from $\tau$. Then it is easy to show that $\mathfrak{M}_\tau$ is invariant in $\mathfrak{M}$. Indeed, $\mathfrak{M}_\sigma \mathfrak{M}_{q_i}$, where $q_i$ is an arbitrary divisor of $(\mathfrak{G})$ from $\tau$ and $\mathfrak{M}_{q_i} \subset \mathfrak{M}_\tau$, is a subgroup of the group $\mathfrak{M}$. From the case just considered it follows that
$$\mathfrak{M}_\sigma \mathfrak{M}_{q_i} = \mathfrak{M}_\sigma \times \mathfrak{M}_{q_i}.$$
From this it is not difficult to see that $\mathfrak{M}_\sigma$ and $\mathfrak{M}_\tau$ are elementwise permutable. Consequently, $\mathfrak{M}_\tau$ is invariant in $\mathfrak{M}$, and, on the basis of case 1,
$$\mathfrak{M}^G \subseteq \mathfrak{H}.$$
A contradiction.

b) The index of $\mathfrak{M}^{(1)}$ in $\mathfrak{M}$ is some prime number from $\sigma$. According to the induction hypothesis, it is easy to show that $\mathfrak{M}$ contains an invariant subgroup $\mathfrak{M}_\tau$. Then from case 1 it follows that

$$\mathfrak{M}^G \subseteq \mathfrak{H}.$$

Again we have a contradiction.

Thus the theorem is proved completely.

From the theorem obtained and the theorem of S. A. Chunikhin and I. K. Chunikhina ([1]) it follows:

**Theorem 2.** *If $\sigma = \{p\}$ and the strong $N_\sigma^\tau$-Sylow theorem and the $\tau$-Sylow theorem hold for $\mathfrak{G}$, then for every subgroup $\mathfrak{M}$, whose order divides $(\mathfrak{H})$, there exists an element $G \in \mathfrak{G}$ such that*

$$\mathfrak{M}^G \subseteq \mathfrak{H}.$$

Applying Theorem 4 of our paper ([2]), it is easy to prove the following assertion:

**Theorem 3.** *If the strong $N_\sigma^\tau$-Sylow theorem and the $\tau$-Sylow theorem hold for $\mathfrak{G}$, then for every $\sigma\Delta$-subgroup $\mathfrak{M}$, whose order divides $(\mathfrak{H})$, there exists an element $G \in \mathfrak{G}$ such that*

$$\mathfrak{M}^G \subseteq \mathfrak{H}.$$

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2. S. A. Rusakov, *DAN*, **127**, No. 2, 270 (1959).
3. S. A. Rusakov, *DAN*, **141**, No. 2, 320 (1961).

*Note: Figure translations are in progress. See original paper for figures.*