# ON THE APPLICATION OF ASYMPTOTIC METHODS TO THE STUDY OF GYROSCOPIC SYSTEMS

Consider the system of equations

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## Abstract

## Full Text

## MECHANICS

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# ON THE APPLICATION OF ASYMPTOTIC METHODS TO THE STUDY OF GYROSCOPIC SYSTEMS

*(Presented by Academician A. Yu. Ishlinskii, 25 V 1962)*

Consider the system of equations

$$\begin{aligned} \ddot{\alpha} + \dot{\beta} &= \varepsilon f_1(\dot{\alpha}, \dot{\beta}, \alpha, \beta, \alpha, \beta); \\ \ddot{\beta} - \dot{\alpha} &= \varepsilon f_2(\dot{\alpha}, \dot{\beta}, \alpha, \beta, \alpha, \beta), \end{aligned} \tag{1}$$

where $\varepsilon$ is a small parameter.

We seek the solution of (1) in the form

$$\begin{aligned} \alpha &= \alpha_0 - a \sin\psi + \varepsilon u_1(a, \psi, \alpha_0, \beta_0) + \varepsilon^2 u_2(a, \psi, \alpha_0, \beta_0) + \cdots, \\ \beta &= \beta_0 + a \cos\psi + \varepsilon v_1(a, \psi, \alpha_0, \beta_0) + \varepsilon^2 v_2(a, \psi, \alpha_0, \beta_0) + \cdots, \end{aligned} \tag{2}$$

where $u_i$ and $v_i$ are periodic functions of $\psi$ with period $2\pi$, and the quantities $\alpha_0, \beta_0, a, \psi$ are determined from the system of equations

$$\begin{aligned} da/dt &= \varepsilon A_1(a, \alpha_0, \beta_0) + \varepsilon^2 A_2(a, \alpha_0, \beta_0) + \cdots, \\ d\psi/dt &= 1 + \varepsilon B_1(a, \alpha_0, \beta_0) + \varepsilon^2 B_2(a, \alpha_0, \beta_0) + \cdots, \\ d\alpha_0/dt &= \varepsilon a_1(a, \alpha_0, \beta_0) + \varepsilon^2 a_2(a, \alpha_0, \beta_0) + \cdots, \\ d\beta_0/dt &= \varepsilon b_1(a, \alpha_0, \beta_0) + \varepsilon^2 b_2(a, \alpha_0, \beta_0) + \cdots. \end{aligned} \tag{3}$$

It has been shown, analogously to ([1]), that for the unambiguous choice of the coefficients of $\varepsilon^k$ in (2) and (3) one may take, as additional conditions, the absence of the zero harmonic in $u_i, v_i$ and the absence of the first harmonic in $v_i$.

Using (2) and (3), one can obtain expressions for $d\alpha/dt$, $d\beta/dt$, $d^2\alpha/dt^2$, $d^2\beta/dt^2$, arranged according to powers of the parameter $\varepsilon$. Using these expressions, as well as expressions (2), we represent the right- and left-hand sides of (1) as series in powers of $\varepsilon$.

Requiring that expression (2) satisfy equations (1) with accuracy up to $\varepsilon^{m+1}$, we equate the coefficients of like powers of $\varepsilon$ in the expansions of the left- and right-hand sides of equations (1) up to order $m$, inclusive. As a result we obtain the following systems of equations:

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial\psi^2} + \frac{\partial v_1}{\partial\psi} &= f_{10} + A_1\cos\psi - aB_1\sin\psi - b_1,\\
\frac{\partial^2 v_1}{\partial\psi^2} - \frac{\partial u_1}{\partial\psi} &= f_{20} + A_1\sin\psi + aB_1\cos\psi + a_1,\\
\frac{\partial^2 u_2}{\partial\psi^2} + \frac{\partial v_2}{\partial\psi} &= f_{11} + A_2\cos\psi - aB_2\sin\psi - b_2,\\
\frac{\partial^2 v_2}{\partial\psi^2} - \frac{\partial u_2}{\partial\psi} &= f_{21} + A_2\sin\psi + aB_2\cos\psi + a_2,
\end{aligned} \tag{4}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where

$$\begin{aligned}
f_{10} = f_1(&a, \sin\psi, -a\cos\psi, -a\cos\psi, -a\sin\psi, \alpha_0 - a\sin\psi, \beta_0 +\\
&+ a\cos\psi),\\
f_{20} = f_2(&a\sin\psi, -a\cos\psi, -a\cos\psi, -a\sin\psi, \alpha_0 - a\sin\psi, \beta_0 +\\
&+ a\cos\psi).
\end{aligned}$$

$$\begin{aligned}
f_{11} = {}& \frac{\partial f_1}{\partial\alpha}\left(-2A_1\cos\psi + 2B_1 a\sin\psi + \frac{\partial^2 u_1}{\partial\psi^2}\right) +\\
&+ \frac{\partial f_1}{\partial\beta}\left(-2A_1\sin\psi - 2B_1 a\cos\psi + \frac{\partial^2 v_1}{\partial\psi^2}\right) +\\
&+ \frac{\partial f_1}{\partial\alpha}\left(a_1 - A_1\sin\psi - aB_1\cos\psi + \frac{\partial u_1}{\partial\psi}\right) +\\
&+ \frac{\partial f_1}{\partial\beta}\left(A_1\cos\psi - aB_1\sin\psi + \frac{\partial v_1}{\partial\psi} + b_1\right) +\\
&+ \frac{\partial f_1}{\partial a}u_1 + \frac{\partial f_1}{\partial\beta}v_1 - b_1\frac{\partial v_1}{\partial\beta_0} - A_1\frac{\partial v_1}{\partial a} - a_1\frac{\partial v_1}{\partial\alpha_0} - B_1\frac{\partial v_1}{\partial\psi} - \frac{\partial a_1}{\partial a}A_1 - \frac{\partial a_1}{\partial\alpha_0}a_1 -\\
&- \frac{\partial a_1}{\partial\beta_0}b_1 + \sin\psi\left[A_1\frac{\partial A_1}{\partial a} + \frac{\partial A_1}{\partial\alpha_0}a_1 + \frac{\partial A_1}{\partial\beta_0}b_1 - aB_1^2\right] +\\
&+ \cos\psi\left[a\left(\frac{\partial B_1}{\partial\alpha_0}a_1 + \frac{\partial B_1}{\partial a}A_1 + \frac{\partial B_1}{\partial\beta_0}b_1\right) + 2A_1B_1\right] -\\
&- 2A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi} - 2a_1\frac{\partial^2 u_1}{\partial\alpha_0\,\partial\psi} - 2b_1\frac{\partial^2 u_1}{\partial\beta_0\,\partial\psi}.
\end{aligned}$$

We have an analogous expression for $f_{21}$. The arguments of $\partial f_1/\partial\alpha$, $\partial f_1/\partial\dot\alpha$, $\partial f_1/\partial\dot\beta, \dots$ in the expressions $f_{11}$ and $f_{21}$ are the same as in $f_{10}$ and $f_{20}$.

Representing in the form of Fourier series

$$f_{10} = g_{10}(a, \alpha_0, \beta_0) + \sum_{n=1}^{\infty} [g_{1n}(a, \alpha_0, \beta_0) \cos n\psi + h_{1n}(a, \alpha_0, \beta_0) \sin n\psi] ,$$
$$f_{20} = g_{20}(a, \alpha_0, \beta_0) + \sum_{n=1}^{\infty} [g_{2n}(a, \alpha_0, \beta_0) \cos n\psi + h_{2n}(a, \alpha_0, \beta_0) \sin n\psi] , \quad (5)$$

$$u_1 = v_0^{\alpha}(a, \alpha_0, \beta_0) + \sum_{n=1}^{\infty} [v_n^{\alpha}(a, \alpha_0, \beta_0) \cos n\psi + w_n^{\alpha}(a, \alpha_0, \beta_0) \sin n\psi] ,$$
$$v_1 = v_0^{\beta}(a, \alpha_0, \beta_0) + \sum_{n=1}^{\infty} [v_n^{\beta}(a, \alpha_0, \beta_0) \cos n\psi + w_n^{\beta}(a, \alpha_0, \beta) \sin n\psi] ,$$

substituting (5) into (4), equating the coefficients of equal harmonics, and recalling the additional conditions indicated above, we find:

$$b_1 = g_{10}, \qquad a_1 = -g_{20}; \quad (6)$$

$$A_1 = -\frac{g_{11} + h_{21}}{2}, \qquad B_1 = \frac{h_{11} - g_{21}}{2a},$$

$$v_1^{\alpha} = \frac{h_{21} - g_{11}}{2}, \qquad w_1^{\alpha} = -\frac{g_{21} + h_{11}}{2}, \qquad v_0^{\alpha} = v_0^{\beta} = v_1^{\beta} = w_1^{\beta} = 0; \quad (7)$$

$$w_n^{\alpha} = \frac{g_{2n} - nh_{1n}}{n(n^2 - 1)}, \qquad v_n^{\alpha} = -\frac{h_{2n} + ng_{1n}}{n(n^2 - 1)}, \quad (8)$$

$$w_n^{\beta} = -\frac{h_{2n}n + g_{1n}}{n(n^2 - 1)}, \qquad v_n^{\beta} = \frac{h_{1n} - ng_{2n}}{n(n^2 - 1)}, \qquad n = 2, 3, ...$$

Approximations of higher orders can be found similarly. As an example, let us consider the equations of motion of a balanced gyroscope in a gimbal suspension on a fixed base, in the presence of small friction in the suspension axes,

$$(B + D)\ddot{\theta} + (B + E - C)\dot{\psi}^2 \cos\theta \sin\theta + A(\dot{\varphi} - \dot{\psi}\sin\theta)\dot{\psi}\cos\theta = M_{\theta},$$

$$\frac{d}{dt}\left\{[F + (B + E)\cos^2\theta + C\sin^2\theta]\dot{\psi} - A(\dot{\varphi} - \dot{\psi}\sin\theta)\sin\theta\right\} = M_{\psi}, \quad (9)$$

$$\frac{d}{dt}\left[A(\dot{\varphi} - \dot{\psi}\sin\theta)\right] = M_{\varphi},$$

where $\Psi$ is the angle of rotation of the outer ring; $\theta$ is the angle of rotation of the inner ring; $\varphi$ is the angle of proper rotation of the rotor; $A$ is the polar moment of inertia of the rotor; $B$ is the equatorial moment of inertia of the rotor; $D, E, C$ are the moments of inertia of the inner ring with respect to the axes $X, Y, Z$, respectively; $F$ is the moment of inertia of the outer ring with respect to its axis of rotation.

We assume

$$M_\varphi = 0, \qquad M_\theta = -l_1\dot{\theta} - L_1 \operatorname{sgn}\dot{\theta}, \qquad M_\Psi = -l_2\dot{\Psi} - L_2 \operatorname{sgn}\dot{\Psi}.$$

Denoting $A(\dot{\varphi} - \dot{\Psi}\sin\theta) = H$, representing $\theta = \theta_0 + \Delta\theta$ and the trigonometric functions of $\theta$ to within small quantities of first order with respect to $\Delta\theta$, introducing the dimensionless time

$$\tau = \frac{H\cos\theta_0\, t}{\sqrt{GI_0}},$$

where

$$G = B + D, \qquad K = B + E - C, \qquad I_0 = F + C + (B + E - C)\cos^2\theta_0,$$

and denoting

$$\frac{K}{\sqrt{GI_0}}\sin 2\theta_0 = c_1,$$

$$\sqrt{\frac{I_0}{G}}\operatorname{tg}\theta_0 = c_2, \qquad \frac{K}{G}\cos 2\theta_0 = c_3,$$

$$\frac{l_1}{H\cos\theta_0}\sqrt{\frac{I_0}{G}} = m_1, \qquad \frac{L_1\sqrt{GI_0}}{H^2\cos^2\theta_0} = M_1,$$

$$\frac{l_2}{H\cos\theta_0}\sqrt{\frac{G}{I_0}} = m_2, \qquad \frac{L_2 G}{H^2\cos^2\theta_0} = M_2,$$

$$\sqrt{\frac{G}{I_0}}\Delta\theta = \alpha, \qquad \Psi = \beta,$$

we bring the equations to the form

$$\begin{gathered}\ddot{\alpha} + \dot{\beta} = \\ = -\frac{c_1}{2}\dot{\beta}^2 + c_2\dot{\alpha}\dot{\beta} - c_3\alpha\dot{\beta}^2 - m_1\dot{\alpha} - M_1\operatorname{sgn}\dot{\alpha}, \\ \ddot{\beta} - \dot{\alpha} = c_1(\alpha\dot{\beta} + \dot{\alpha}\dot{\beta}) - c_2\alpha\dot{\alpha} + \\ +2c_3\alpha\dot{\alpha}\dot{\beta} - m_2\dot{\beta} - M_2\operatorname{sgn}\dot{\beta}. \qquad (10)\end{gathered}$$

**Fig. 1**

Fig. 1

Figure 1: Fig. 1

Considering the right-hand sides of (10) as $\varepsilon f_1$ and $\varepsilon f_2$, we obtain that system (3) in the first approximation assumes the form

$$\frac{d\alpha_0}{d\tau} = 0, \qquad \frac{d\beta_0}{d\tau} = \left(-\frac{c_1}{4} + \frac{c_2}{2} - \frac{c_3}{2}\alpha_0\right) a^2, \tag{11}$$

$$\frac{da}{d\tau} = -\frac{a}{2}(m_1 + m_2) - \frac{2}{\pi}(M_1 + M_2), \qquad \frac{d\psi}{d\tau} = 1 + \frac{c_1}{2}\alpha_0 - c_2\alpha_0 + \frac{5}{8}c_3 a^2,$$

in which it is not difficult to integrate it. In the absence of friction, from the second equation of system (11) we obtain a constant drift of the gyroscope (2). The same has been done not under the assumption $M_\varphi = 0$, but under the assumption $\dot{\varphi} = \text{const}$.

In conclusion, we make a few remarks. To exclude special solutions of (1) from consideration, appropriate restrictions must be imposed on the dependence of $f_1$ and $f_2$ on the higher derivatives. As for possible generalizations of what has been set forth above and questions connected with mathematical justification, here one can also use the ideas set forth in (1). The first approximation, as always, is equivalent to the averaging method (3).

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*Note: Figure translations are in progress. See original paper for figures.*