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Abstract

Full Text

MATHEMATICS

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ON RECURSIVE ABELIAN GROUPS

Adhering to the terminology of the survey ([1]), we shall henceforth call a group G, equipped with a one-to-one mapping α of some recursive set D_α of natural numbers onto G, a constructive (or recursive) group, provided that there exist recursive functions $\theta(x, y),\ f(x, y)$ having the properties:

$$\theta(x, y) = 1 \leftrightarrow \alpha x = \alpha y, \qquad \alpha f(x, y) = \alpha x \cdot \alpha y \quad (x, y \in D_\alpha).$$

A mapping α having all the properties listed is called a constructive numbering of the group G. Groups for which constructive numberings exist are called constructivizable or computable. A general problem naturally arises: to determine which constructive numberings are admitted by one or another abstractly given group, which subgroups of one or another constructive group are its recursive or recursively enumerable subgroups, and so on. Below we indicate some initial results in this direction for the case of abelian groups.

1. From the remarks in the survey ([1]) it follows that, in proving the "positive" parts of the theorems below, one may assume that the elements of a constructive group are natural numbers, and that the group operation is a general recursive binary function. Examples of constructive groups will be given in the form of groups with defining relations. In this case, if the generators of the group are the symbols $x_1, x_2, \ldots$, then the element with number n will always be assumed to be the element $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $n_1, n_2, \ldots, n_k$ is the sequence of integers with number n in the standard numbering (see ([1])).

Theorem 1. *The periodic part of a constructive group is recursively enumerable. There exist constructive abelian groups whose periodic part is not recursive.*

The algorithm for enumerating the periodic part is constructed in an obvious way. As a constructive group with nonrecursive periodic part one may take the abelian group with generators $x_1, x_2, \ldots$ and defining relations

$$x_{\lambda(n)}^n = 1 \quad (n = 1, 2, \ldots),$$

where $\lambda(n)$ is a one-valued general recursive function whose set of values is not recursive.

If G is a group, then by $\Pi(G)$ we denote the set of those primes p for which there exists in G an element of order p. The p-primary component of G is the set of elements $G_{(p)}$, including the identity of G, whose orders have the form p^s $(s = 1, 2, ...)$.

Theorem 2. *All primary components of a constructive periodic group G are recursive; the set $\Pi(G)$ is recursively enumerable. For every recursively enumerable set Π of primes there exists a periodic abelian constructive group G for which $\Pi = \Pi(G)$.*

Let $G_1, G_2, ...$ be some sequence of numbered groups, and let α_n be a numbering of G_n with numbered set D_n. Assigning to the natural number n the element $\alpha_1 n_1 \cdot \alpha_2 n_2 \cdot ... \cdot \alpha_k n_k$ of the direct product $G = G_1 \times G_2 \times ...$, where $n_1, ..., n_k$ is the set of natural numbers having standard number n, we obtain a numbering of the group G, which we shall call **standard**.

If the characteristic function $\chi(n, x)$ of the set D_n, and also the functions $\theta_n(x, y)$ and $f_n(x, y)$ for the group G_n, are general recursive functions of x, y, n, then the standardly numbered direct product $G_1 \times G_2 \times ...$ will be a constructive group. In particular, if G is a constructive periodic abelian group and $\Pi(G) = \{p_{\nu(1)}, p_{\nu(2)}, ...\}$, where $\nu(i)$ is a one-one recursive function, then G is constructively equivalent to the direct product of its primary subgroups $G_{p_{\nu(1)}} G_{p_{\nu(2)}}, ...$, taken with the standard numbering.

2. We shall call numberings α and β of some algebra $\mathfrak{A}$ **autoequivalent** if there exists an abstract automorphism σ of the algebra $\mathfrak{A}$ such that the numberings $\sigma\alpha$ and β are recursively equivalent. We shall call an algebra $\mathfrak{A}$ **autostable** if all its constructive numberings are autoequivalent. Algebras all of whose constructive numberings are recursively equivalent were called recursively stable in note ([2]). Recursive stability obviously implies autostability, but not conversely. Each abstractly given algebra has at most countably many constructively equivalent numberings. Automorphisms transform constructive numberings into constructive ones. Therefore an algebra with an uncountable group of automorphisms cannot be recursively stable.

An abelian group A_{pn} with generators a_{ij} and defining relations $a_{ij+1}^p = a_{ij}$ (p prime, $i = 1, ..., n;\ j = 0, 1, ...;\ a_{i0} = 1$) is called a complete p-primitive abelian group of rank n. Its standard numbering is constructive. The automorphism group of A_{pn} is uncountable, and therefore A_{np} is not recursively stable. At the same time it is easy to prove that A_{np}, for any finite n, is an autostable group.

It is known that complete torsion-free abelian groups of finite rank are computable and recursively stable ([2]). A complete torsion-free abelian group R_∞ of countable rank is the direct product of groups of rank 1 and therefore admits constructive numberings. The automorphism group of R_∞ is uncountable, and therefore R_∞ is not recursively stable.

We shall call a constructive abelian group G a group with an algorithm of linear dependence if there exists an algorithm that, for any system of natural numbers $n_1, n_2, \ldots, n_k$, makes it possible to say whether or not the elements with numbers $n_1, n_2, \ldots, n_k$ are linearly dependent in G.

Theorem 3. In order that a constructive torsion-free abelian group of countable rank have an algorithm of linear dependence, it is necessary and sufficient that it have a recursively enumerable basis. A constructive numbering α of the group R_∞ is autoequivalent to a constructive numbering β of the group R_∞ having a recursively enumerable basis if and only if α also has a recursively enumerable basis. There exist constructive numberings of R_∞ under which there is no recursively enumerable basis in R_∞.

To construct an example of a constructive complete torsion-free abelian group that has no algorithm of linear dependence, denote by R_∞ the additive group of linear forms in variables $x_1, x_2, \ldots$ with rational coefficients and with the standard numbering of these forms. Let $\nu(n)$ be a recursive one-one function whose set of values is not recursive. The subspace H, generated in R_∞ by the forms $ix_{2\nu(i)} - x_{2\nu(i)+1}$ $(i = 0, 2, \ldots)$, is, as is easy to see, recursive. Therefore the numbering of R_∞ is a constructive numbering of the factor group R_∞/H. There is no algorithm of linear dependence in R_∞/H, since the question of linear–

of the dependence of x_{2n} and x_{2n+1} in $R_\infty/\bar{H}$ is equivalent to the question of whether the number n belongs to the range of the function $\nu(i)$, which, by assumption, is not algorithmically decidable.

From Theorem 3 it follows, in particular, that an Abelian complete group without torsion of countable rank is computable, but is not autostable.

The indicated numbering of the group $R_\infty/\bar{H}$ also gives a negative answer to the question of the existence of an algorithm that would make it possible, in every constructive Abelian group of rank 2 without torsion, to find a pair of basis elements. More precisely: let $U(n, x, y)$ be Kleene' s universal partial recursive function, which, for different values of n, gives all possible two-place partial recursive functions. The question is whether there exist general recursive functions $\varphi(n)$, $\psi(n)$ such that, if for some n the function $U(n, x, y)$ is a group operation on the set of natural numbers, turning this set into a complete Abelian group without torsion of rank 2, then the numbers $\varphi(n)$, $\psi(n)$ will be linearly independent elements of this group? Theorem 3 yields a negative answer to this question.

3. Let G be a constructive Abelian group without torsion of rank r, with given basis elements $g_1, \ldots, g_r$. Finding, for each $g \in G$, integers $m, m_1, \ldots, m_r$ such that $mg = m_1 g_1 + \cdots + m_{rg} r$, and assigning to the element g the linear form $\frac{m_1}{m} + \frac{m_2}{m} x_2 + \cdots + \frac{m_r}{m} x_r$ in the independent variables $x_2, \ldots, x_r$, we obtain a recursive mapping of G into the group of linear forms R_r. Therefore every constructive Abelian group without torsion of rank r is recursively equivalent to a suitable recursively enumerable subgroup of the group R_r containing the unit forms $1, x_2, \ldots, x_r$. The same is also true for the group R_∞ with the natural numbering

and for constructive Abelian groups G without torsion with a recursively enumerable basis.

We now want to consider in greater detail subgroups of the group R_1. Let G be a subgroup of the group R_1 containing the number 1. Denote by $D(G)$ the collection of pairs (i, n) of natural numbers for which $p_i^{-n} \in G$, where p_i is the i-th prime.

Lemma. A subgroup $G \subseteq R_1$ is recursive or recursively enumerable if and only if, respectively, the set $D(G)$ is recursive or recursively enumerable.

In group theory, subgroups of R_1 containing 1 are usually described by means of characteristics, i.e. sequences of the form $\alpha(G) = (\alpha_0, \alpha_1, \alpha_2, ...)$, where each α_i is either a natural number or the symbol ∞. The passage from $D(G)$ to $\alpha(G)$ is made according to the rule: $\alpha_i = n$ if $p_i^{-n} \in G$, $p_i^{-n-1} \notin G$, and $\alpha_i = \infty$ if $p_i^{-n} \in G$ for $n = 1, 2, ...$

Introduce the partial function $\alpha(x)$, taking $\alpha(i)$ to be undefined if $\alpha_i = \infty$, and taking $\alpha(i) = \alpha_i$ for all other values of i. The function $\alpha(x)$, as well as the sequence α, will henceforth be called the characteristic of the subgroup G. The connection between $D(G)$ and the corresponding characteristic $\alpha(x)$ is expressed by the formula

$$\alpha(i) = \mu x((i, x) \in D(G)) - 1,$$

from which it follows easily:

Theorem 4. A subgroup G is recursive if and only if its characteristic is representable in the form

$$\alpha(i) = \mu x(f(i, x) = 0),$$

where $f(i, x)$ is a suitable general recursive function.

A subgroup G is recursively enumerable if and only if its characteristic is representable in the form

$$\alpha(i) = \mu x(f(i, x) = \text{undefined}),$$

where $f(i, x)$ is a suitable partial recursive function.

We shall call the number n an ordinary point of the subgroup G, if $a_n \neq \infty$. The set of ordinary points of a recursive subgroup is a recursively enumerable set. Every recursively enumerable set of natural numbers is the set of ordinary points of a suitable recursive subgroup.

Indeed, if $f(x)$ is some recursive function, then the subgroup G with characteristic

$$\alpha(i) = \mu x\,(f(x) = i)$$

is, by Theorem 4, recursive and has as its set of ordinary points the set of values of the function f.

Remark. Let $\alpha(x)$ be the characteristic of some subgroup G of the group R_1. The set $\alpha^{(n)}$ of those x for which $\alpha(x) \geq n$ or $\alpha(x)$ is undefined is recursive for recursive G and recursively enumerable for recursively enumerable G.

As an example, consider the subgroup G with characteristic $\alpha(i) = \mu x\,(U(i,x) \text{ is undefined})$, where $U(i,x)$ is Kleene' s universal function. The set $\alpha^{(n)}$ here is the set of numbers of those functions that are defined at the points $0, 1, \ldots, n-1$. By Rice' s theorem $\alpha^{(n)}$ cannot be recursive. By Theorem 4 and the remark, G is a nonrecursive recursively enumerable subgroup in R_1.

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REFERENCES

[1] A. I. Mal' tsev, UMN, **16**, No. 3 (1961). [2] A. I. Mal' tsev, DAN, **145**, No. 2 (1962).

Note: Figure translations are in progress. See original paper for figures.