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Abstract

Full Text

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On the Theory of Wave Operators and Scattering Operators

(Presented by Academician V. I. Smirnov on 5 I 1962)

In the present communication the concept of wave operators is extended to the case of a pair of unitary operators. The existence of these wave operators is established for unitary operators that differ by a trace-class operator. The use of the Cayley transform then makes it possible to obtain the existence of wave operators for a pair of self-adjoint operators under the single condition that the difference of the resolvents be trace class*.

By means of the wave operators, in the usual way, one constructs the scattering operator, which in turn gives rise to the S -matrix (the scattering matrix). In the second part of the communication, some spectral characteristics of the scattering matrix are established.

1. Let U_1, U_2 be unitary operators in a Hilbert space \mathfrak{H} , and let P_k ($k = 1, 2$) be the projector onto the absolutely continuous subspace $\mathfrak{H}_k \subset \mathfrak{H}$ of the operator U_k .

Theorem 1. *If the operator $V = U_2 - U_1$ is trace class, then:*

- 1) *there exist the strong limits (wave operators)*

$$W_{\pm}(U_2, U_1) = \lim_{n \rightarrow \pm\infty} U_2^n U_1^{-n} P_1; \quad (1)$$

$$W_{\pm}(U_1, U_2) = \lim_{n \rightarrow \pm\infty} U_1^n U_2^{-n} P_2. \quad (2)$$

- 2) *the operators $W_{\pm}(U_2, U_1)$ isometrically map \mathfrak{H}_1 onto \mathfrak{H}_2 ; the operators $W_{\pm}(U_1, U_2) = W_{\pm}^*(U_2, U_1)$ isometrically map \mathfrak{H}_2 onto \mathfrak{H}_1 .*
- 3) *The absolutely continuous parts of the operators U_1 and U_2 are unitarily equivalent, and for every $f \in \mathfrak{H}_1$*

$$W_{\pm}(U_2, U_1)U_1 f = U_2 W_{\pm}(U_2, U_1) f.$$

The proof of this theorem is carried out essentially by the method of T. Kato^(2,3). Namely, in the case of a one-dimensional perturbation the proof is based on the study of the resolvents of the operators U_1, U_2 . The passage to the case

of finite-dimensional V is made on the basis of the “multiplication theorem” for wave operators ^(2,5). Finally, in the general case a limiting passage from finite-dimensional perturbations is used. In this connection we note that the approximation of a perturbation by simpler operators V_n must be carried out without destroying the unitarity of the operator $U_1 + V_n$.

* Wave operators for abstract self-adjoint operators were studied in ⁽¹⁻⁶⁾. A brief comparison of the results is given in the note ⁽⁶⁾; see there also the definition of a wave operator, of the absolutely continuous part of an operator, etc. In contrast to note ⁽⁶⁾, we denote the wave operators by W_{\pm} , and not by U_{\pm} . Concerning the definition of the class of trace-class operators, see, for example, ⁽⁷⁾. In the note ⁽⁹⁾ of one of the authors this class was denoted by \mathfrak{S} . The condition that the difference of the resolvents $R_z(H_2), R_z(H_1)$ be trace class means that for some common regular point z (and then for every such point) $R_z(H_2) - R_z(H_1) \in \mathfrak{S}$.

The possibility of such an approximation was earlier indicated by one of the authors ⁽⁸⁾ in extending the “trace formulas” to unitary operators.

2. Let now H_1, H_2 be self-adjoint operators in \mathfrak{H} . We shall say that the operators H_1, H_2 satisfy condition (A) if the difference of their resolvents is nuclear. Let $n(> 1)$ be an integer; we shall say that the operators H_1, H_2 satisfy condition (B_n) if they are positive definite, the operator $H_2^{-1} - H_1^{-1}$ is completely continuous, and the operator $H_2^{-n} - H_1^{-n}$ is nuclear. For the projection onto the absolutely continuous subspace \mathfrak{H}_k of the operator H_k we retain the notation P_k ($k = 1, 2$).

Theorem 2. *Let H_1, H_2 be self-adjoint operators in \mathfrak{H} satisfying condition (A). Then there exist the strong limits*

$$W_{\pm}(H_2, H_1) = \lim_{t \rightarrow \pm\infty} e^{iH_2 t} e^{-iH_1 t} P_1. \quad (3)$$

If $U_k = (H_k - iI)(H_k + iI)^{-1}$, ($k = 1, 2$), are the Cayley transforms of the operators H_k , then the difference $U_2 - U_1$ is nuclear and

$$W_{\pm}(H_2, H_1) = W_{\pm}(U_1, U_2). \quad (4)$$

Obviously, assertions analogous to assertions 2) and 3) of Theorem 1 are also valid. We note that Kato’s theorem ⁽³⁾ on perturbations of a self-adjoint operator by a nuclear one follows already from Theorem 1 (it suffices to apply Theorem 1 to the operators $\exp(iH_k)$). Both of Kuroda’s criteria ⁽⁴⁾ for the existence of wave operators are contained in Theorem 2. In proving Theorem 2 it is enough to establish the existence of the weak limits (3), as well as relation (4). Then, by Theorem 1, the operators $W_{\pm}(H_2, H_1)$ are isometric and the limits (3) are strong. By this method the existence of the strong limits (3) was earlier

proved by one of the authors ⁽⁶⁾ under the additional condition of bounded invertibility of the operators H_1, H_2 . Instead of formula (4), in that case the relation $W_{\pm}(H_2, H_1) = W_{\mp}(H_2^{-1}, H_1^{-1})$ was established. In the same work the existence of strong limits of the form (3) was proved for the operators H_1, H_2 and the operators H_1^{-1}, H_2^{-1} under condition (B_n) . It was shown there that

$$W_{\pm}(H_2, H_1) = W_{\mp}(H_2^{-1}, H_1^{-1}) = W_{\mp}(H_2^{-n}, H_1^{-n}). \quad (5)$$

3. With the aid of the wave operators one constructs the scattering operator

$$S \equiv S_{21} \equiv S(H_2, H_1) = W_+^*(H_2, H_1) W_-(H_2, H_1),$$

widely used in physics. The operator S is unitary in the space \mathfrak{H}_1 and commutes there with H_1 . From the multiplication theorem for wave operators there easily follows the multiplication rule for scattering operators, which we give for the case of three factors:

$$S(H_4, H_1) \equiv S_{41} = S'_{43} S'_{32} S_{21}, \quad (6)$$

where

$$S'_{k+1,k} = W_+^*(H_k, H_1) S(H_{k+1}, H_k) W_+(H_k, H_1) \quad (k = 2, 3)$$

are unitary in \mathfrak{H}_1 operators commuting with H_1 . Suppose \mathfrak{H}_1 is decomposed ⁽⁹⁾ into a continuous direct sum of Hilbert spaces \mathfrak{h}_{λ} over the ring of bounded functions of the absolutely continuous part of the operator H_1 . Since all spectral measures of the operator H_1 in \mathfrak{H}_1 are absolutely continuous, we may assume that the measure associated with the decomposition of \mathfrak{H}_1 into the sum of the spaces \mathfrak{h}_{λ} is Lebesgue measure on the spectrum Λ of the operator

* Without restricting generality, we shall regard \mathfrak{H}_1 as separable, since for perturbations of the types considered

H_1 in \mathfrak{H}_1 . Since the operator S commutes with H_1 , it gives rise, in the family \mathfrak{H}_{λ} , to a measurable family of unitary operators $S(\lambda)$, defined for almost all $\lambda \in \Lambda$. We shall call the family $S(\lambda)$ the **scattering matrix** corresponding to the operator S . If the multiplicity of the spectrum of H_1 in \mathfrak{H}_1 is finite, then all \mathfrak{H}_{λ} are finite-dimensional, and $S(\lambda)$ can be realized in the form of a finite unitary matrix-function. In the contrary case, $S(\lambda)$ can be realized (depending on the choice of realization of the spaces \mathfrak{H}_{λ}) in the form of infinite matrix-functions or kernels. The multiplication rule (6) is, evidently, carried over also to scattering matrices:

$$S_{41}(\lambda) = S_{43}(\lambda) S_{32}(\lambda) S_{21}(\lambda). \quad (7)$$

4. In proving the results presented below, an essential role is played by the apparatus introduced to justify the trace formula ^(10,8). We give the information needed for what follows. If condition (A) is satisfied, then for the trace of the difference of the resolvents of the operators H_1, H_2 , for any non-real z , the representation

$$\text{Sp}\{R_z(H_2) - R_z(H_1)\} = - \int_{-\infty}^{\infty} \frac{\xi(\lambda) d\lambda}{(\lambda - z)^2}, \quad (8)$$

is valid, where $(1 + \lambda^2)^{-1}\xi(\lambda) \in L_1(-\infty, \infty)$. The spectral shift function $\xi(\lambda)$ is determined by relation (8) up to a constant term. In what follows it is sufficient for us to determine $\xi(\lambda)$ up to an integer summand by means of the relation

$$\int_{-\infty}^{\infty} \frac{\xi(\lambda) d\lambda}{1 + \lambda^2} = \frac{1}{2i} \text{Sp} \ln(U_1^{-1}U_2), \quad (9)$$

where U_k ($k = 1, 2$) are the Cayley transforms of the operator H_k . If $H_2 = H_1 + V$, where V is a nuclear operator, then $\xi(\lambda) \in L_1(-\infty, \infty)$, and condition (9) can be replaced by the condition

$$\int_{-\infty}^{\infty} \xi(\lambda) d\lambda = \text{Sp} V.$$

If V is one-dimensional, then the signs of $\xi(\lambda)$ and V are the same and $|\xi(\lambda)| \leq 1$. The spectral shift function $\xi(\lambda)$ can also be introduced under condition (B_n) (see ⁽⁸⁾) by transforming the independent variable in the spectral shift function for the operators H_1^{-n}, H_2^{-n} . In this case $\xi(\lambda) = 0$ to the left of the spectra of H_1 and H_2 , and $\lambda^{-(n+1)}\xi(\lambda) \in L_1(0, \infty)$.

5. **Theorem 3.** *If, for the self-adjoint operators H_1, H_2 , condition (A) or some condition (B_n) ($n = 2, 3, \dots$) is satisfied, then the scattering matrix for almost all $\lambda \in \Lambda$ has the form*

$$S(\lambda) = I_\lambda + T(\lambda), \quad (10)$$

where I_λ is the identity operator and $T(\lambda)$ is a nuclear operator in \mathfrak{H}_λ . Moreover, for almost all $\lambda \in \Lambda$

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda)}, \quad (11)$$

where $\xi(\lambda)$ is the spectral shift function for H_1 and H_2 .

Let us make some comments. If $H_2 = H_1 + V$ and V is nuclear, then the passage from H_1 to H_2 can be carried out in steps, introducing at each step a one-dimensional perturbation. Such a perturbation corresponds to a scattering

matrix whose spectrum consists of unity and a simple eigenvalue $\exp[-2\pi i\xi_k(\lambda)]$, if the latter is different from unity. Here $\xi_k(\lambda)$ is the spectral shift function corresponding to the introduction of a one-

measurable perturbation at the k -th step. Note that (see ⁽¹⁰⁾)

$$\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} |\xi_k(\lambda)| d\lambda \leq \text{Sp}|V|. \quad (12)$$

Using estimate (12), one can prove that the infinite product of scattering matrices corresponding to the process described above, constructed according to rule (7), converges, after subtracting I_λ from it, in the trace norm for almost all $\lambda \in \Lambda$ to some operator $T(\lambda)$. It is also easily verified that $I_\lambda + T(\lambda)$ is the scattering matrix corresponding to the operator $S(H_2, H_1)$. In the case of condition (A) (condition (B_n)), formula (4) (respectively formula (5)) makes it possible to reduce our problem to the analogous problem for the Cayley transforms U_1, U_2 (respectively for the operators H_1^{-n}, H_2^{-n}), after which one can carry out an argument similar to that presented above.

From (10) there follows the possibility of representing $S(\lambda)$ in the form

$$S(\lambda) = e^{-2\pi i K(\lambda)} \quad (\text{for almost all } \lambda \in \Lambda), \quad (13)$$

where $K(\lambda)$ is a trace-class operator in \mathfrak{h}_λ . $K(\lambda)$ can be chosen so that

$$\int_{\Lambda} \text{Sp}|K(\lambda)| p(\lambda) d\lambda \leq C (< +\infty).$$

Here: 1) $p(\lambda) = 1$, $C = \text{Sp}|V|$, if $H_2 = H_1 + V$ and V is a trace-class operator; 2) $p(\lambda) = (1 + \lambda^2)^{-1}$, $C = \text{Sp}|U_2 - U_1|$, if condition (A) is satisfied; 3) $p(\lambda) = n\lambda^{-(n+1)}$, $C = \text{Sp}|H_2^{-n} - H_1^{-n}|$, if condition (B_n) is satisfied.

It can be shown that the trace-class operator $K(\lambda)$ in representation (13) can be chosen nonnegative, in general under the same conditions under which it was possible in ⁽¹⁰⁾ to establish the lower semiboundedness of the function $\xi(\lambda)$. Namely, the following is true:

Theorem 4. *The trace-class operator $K(\lambda)$ in representation (13) can be chosen nonnegative if condition (A) is satisfied, and also the following conditions: 1) $\mathfrak{D}(H_1) = \mathfrak{D}(H_2)$; 2) the form $((H_2 - H_1)f, f)$ ($f \in \mathfrak{D}(H_1)$) has a finite number of negative squares; there exist constants α, β ($0 \leq \alpha < 1$) such that $\|(H_2 - H_1)f\| \leq \alpha\|H_1 f\| + \beta\|f\|$ for all $f \in \mathfrak{D}(H_1)$.*

It is also easy to see that in representation (13) the trace-class operator $K(\lambda)$ can be chosen nonnegative if, for some n (≥ 1), condition (B_n) is satisfied and the form $((H_1^{-n} - H_2^{-n})f, f)$ has a finite number of negative squares.

6. Let us note in conclusion that, for the radial Schrödinger equation, a rigorous proof of the coincidence of the function $-\pi\xi(\lambda)$ with the limiting phase is contained in the work ⁽¹²⁾. In the three-dimensional problem of quantum scattering, an expression of $\xi(\lambda)$ in terms of the scattering amplitude was recently obtained by V. S. Buslaev ⁽¹³⁾. V. S. Buslaev's formula is interesting in that it is considerably simpler than the expression obtained from the general formula (11).

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* The last condition arose, for other reasons, in the works ^(11,4).

Note: Figure translations are in progress. See original paper for figures.

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