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Soviet-era science, translated into English

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1962

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**Abstract**

**Full Text**

**V. A. Manevich**

**A GENERALIZED CAYLEY PROBLEM AND RELATED QUESTIONS**

*(Presented by Academician P. S. Aleksandrov on 16 XII 1961)*

In 1873 A. Cayley posed the following problem: *Given two tetrahedra  $ABCD$  and  $A'B'C'D'$ . Find the locus of points  $P$  and  $P'$  such that the pencils  $P(PA, PB, PC, PD, \dots)$  and  $P'(P'A', P'B', P'C', P'D', \dots)$  are congruent <sup>(1)</sup>.*

Many eminent geometers worked on the solution of this problem (see <sup>(2)</sup>, p. 428). In 1909 R. Sturm showed that the solution of A. Cayley's problem is connected with the construction of spheroidal collineations <sup>(3)</sup>, but this construction was not obtained.

In the present paper two theorems are proved and two problems are solved, from which, as consequences, follow the construction of a spheroidal collineation and the solution of some other problems.

**Theorem 1.** *Given two triples of points  $A, B, C$  and  $A', B', C'$ , and two conic sections  $\alpha^2$  and  $\alpha\alpha'^2$ , not lying in the planes  $ABC$  and  $A'B'C'$ . Then in space there is established a point correspondence in which to each point  $M \notin \text{pl } ABC$  there correspond 8 points  $M'$  such that the pencil*

$$M(A, B, C, \alpha^2, \dots) \bar{\cap} M'(A', B', C', \alpha\alpha'^2, \dots).$$

The basic idea of the proof is as follows:

$$\alpha\alpha'^2 \cap \text{pl } A'B'C' \equiv (D', E').$$

In the plane  $ABC$  a collineation  $\Gamma : D \rightarrow E$  is established such that for any pair of corresponding points  $D, E$  we have

$$(ABCDE \dots) \bar{\cap} (A'B'C'D'E' \dots).$$

To each point  $M$  of space there corresponds the curve

$$\alpha_1^2 \equiv \text{cone } (M\alpha^2) \cap \text{pl } ABC.$$

In the plane  $A'B'C'$  there exist 4 curves  $\alpha_1'^2$  passing through  $D', E'$  and satisfying the condition

$$(A, B, C, \alpha_1^2, \dots) \bar{\wedge} (A', B', C', \alpha_1'^2, \dots).$$

There exist two cones of the 2nd order passing through  $\alpha a'^2$  and  $\alpha_1'^2$ , whose vertices  $M'_1$  and  $M'_2$  are the desired ones.

Analogous arguments make it possible to solve the following problem:

**Problem 1.** Find the locus of points  $M'$  and  $M$  such that the pencils  $\{M\}$  and  $\{M'\}$ , projecting respectively the figures  $[A, B, C, D, \alpha^2]$  and  $(A', B', C', D', \alpha a'^2)$ , where  $ABCD$  and  $A'B'C'D'$  are plane quadrilaterals, are collinear. The desired loci are curves of the 2nd order. If  $\alpha^2 \equiv \alpha a'^2$  is the absolute curve of space, we obtain Möbius' theorem on bringing two collinear fields into perspective position and the locus of centers of perspectivity.

**Theorem 2.** Given a tetrahedron  $ABCD$ , a plane quadrilateral  $A_1B_1C_1D_1$ , and two curves of the 2nd order  $\alpha^2$  and  $\alpha_1^2$ . Then there is a spatial-

an curve of the 16th order, to each point  $M$  of which there corresponds a point  $M'$  such that the bundles  $\{M\}$  and  $\{M'\}$ , projecting the figures  $(ABCD\alpha^2)$  and  $(A_1B_1C_1D_1\alpha_1^2)$ , are collinear.

The idea of the proof is as follows:  $a_1^2 \cap \text{pl } A_1B_1C_1 \equiv (F_1, G_1)$ . Let  $F$  be an arbitrary point of the plane  $ABC$ . In the collineation  $A_1B_1C_1F_1 \rightarrow ABCF$ , the points  $G_1$  and  $D_1$  pass respectively into  $G$  and  $D'$ . We shall seek such points  $M$  for which  $MD \supset D'$ , and  $MG$  and  $MF$  intersect  $a^2$ . To each line  $DD'$  there corresponds a cone of the 2nd order  $(Ga^2)$ , intersecting  $DD'$  in two points lying on the surface  $\varphi^4$  of the 4th order. On the other hand, to the line  $DD'$  there corresponds a cone of the 2nd order  $(Fa^2)$ ; this correspondence generates a surface  $f^4$  of the 4th order,  $(\varphi^4 \cap f^4)$  is the required curve.

In the case when  $a^2 \equiv a_1^2$  is the absolute curve of space, we obtain a solution of the first problem of N. F. Chetverukhin <sup>(4)</sup> and the Pohlke–Schwarz theorem.

**Problem 2.** Two tetrahedra  $ABCD$ ,  $A'B'C'D'$  and two conic sections  $a^2$  and  $a'^2$  are given. Construct a collineation in which the given tetrahedra would be corresponding, and to some surfaces of the 2nd order passing through  $a^2$  there would correspond surfaces of the 2nd order passing through  $a'^2$ .

The idea of the solution is as follows: we seek a plane  $\alpha_1$ , corresponding to the plane  $\alpha \supset a^2$ , such that the conic section  $a_1^2$ , corresponding to  $a^2$  in the collineation

$$(A, B, C, D, \alpha, \dots) \rightarrow (A', B', C', D', \alpha_1 \dots),$$

intersects  $a'^2$  in two points. The cone  $(D'a'^2) \cap \text{pl } A'B'C' \equiv a_1'^2$ ,  $\alpha_1 \equiv \alpha' \cap \text{pl } A'B'C'$  ( $\alpha'$  is the plane of the curve  $a'^2$ );  $\text{pl } ABC \cap \alpha \equiv a$ ;  $DA, DB, DC$  intersect  $\text{pl } \alpha$  respectively at the points  $A_1, B_1, C_1$ .

Problem 2 is reduced to the following:

Find such a line  $a' \subset \text{pl } A'B'C'$  for which the collineation

$$(A_1, B_1, C_1, a_1, \dots) \rightarrow (A', B', C', a', \dots) \quad (1)$$

takes the conic section  $a^2$  into a conic section  $\varphi^2$  intersecting  $a_1'^2$  at points  $K, L$  such that  $KL \cap a' \equiv M \subset a_1$ .

It has been shown that for each point  $M \subset a_1$  there exist 12 lines  $a'$  satisfying condition (1). Thus, there exists  $\infty^1$  collineations satisfying the requirement of Problem 2.

In the case when  $a^2 \equiv a'^2$  is the absolute curve of space, we obtain spheroidal collineations, thereby resolving the question of bringing two given tetrahedra into perspective position.

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Received  
8 XII 1961

## CITED LITERATURE

<sup>1</sup> A. Cayley, Proc. London Math. Soc., **4**, 396 (1873). <sup>2</sup> Encyclopädie d. mathematischen Wissenschaften, **3**, H. 3–4, 1907. <sup>3</sup> R. Sturm, Die Lehre von den Geometrischen Verwandtschaften, **3**, Leipzig u. Berlin, 1909, p. 211. <sup>4</sup> N. F. Chetverukhin, Tr. Gruz. polytechn. inst., No. 2 (63), 5 (1959).

*Note: Figure translations are in progress. See original paper for figures.*

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