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Abstract

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DETERMINATION OF STRICT INEQUALITIES BETWEEN THE MODULI OF ROOTS IN THE PROCESS OF TRANSFORMING ALGEBRAIC EQUATIONS BY THE LOBACHEVSKY-GRAEFFE METHOD

(Presented by Academician S. L. Sobolev on 28 V 1962)

Let a polynomial with complex coefficients be given,

$$f(z) = \sum_{i=0}^{n} a_i z^i, \qquad a_0 \neq 0, \tag{1}$$

whose roots are arranged in increasing order of their moduli

$$0 < |z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_n|. \tag{2}$$

Construct the polynomial

$$f_k(z) = \prod_{j=0}^{k-1} f\left(\sqrt[k]{z}\, e^{\frac{2\pi}{k}\sqrt{-1}}\right) = a_0^{(k)} + a_1^{(k)} z + \cdots + a_n^{(k)} z^n = 0, \tag{3}$$

whose roots are equal to the k-th powers of the roots of the original polynomial. The coefficients of the functions (3) are transformed according to the formulas proposed in ([1],[2]).

In the present note the following theorem will be proved:

Theorem. *For the inequality $|z_m| < |z_{m+1}|$ between the moduli of the roots of polynomial (1) to hold, it is necessary and sufficient that, for every pair of values χ and ν,*

$$\lim_{k\to\infty} \frac{\left|a_{m-\chi}^{(k)}\right|^{\nu} \left|a_{m+\nu}^{(k)}\right|^{\chi}}{\left|a_m^{(k)}\right|^{\chi+\nu}} = 0, \tag{4}$$

where $1 \leqslant \chi \leqslant m,\ 1 \leqslant \nu \leqslant n-m.$

Conditions (4) are necessary and sufficient. Only the necessary conditions were indicated by Pólya ([3]). Sufficient conditions were found by Valiron ([4]), and then refined and extended by Ostrowski to a broader class of functions ([5]). In practice, the application of these criteria is connected with the cumbersome construction of special majorizing polynomials for each of the transformed polynomials (3), the so-called Newton diagrams.

Proof. Let

$$|z_m| < |z_{m+1}|; \tag{5}$$

then

$$\frac{\left(a_{m-\chi}^{(k)}\right)^{\nu}\left(a_{m+\nu}^{(k)}\right)^{\chi}}{\left(a_{m}^{(k)}\right)^{\chi+\nu}} = \frac{\left[(-1)^{m-\chi}S\left(z_1^{-k}\dots z_{m-\chi}^{-k}\right)\right]^{\nu}\left[(-1)^{m+\nu}S\left(z_1^{-k}\dots z_{m+\nu}^{-k}\right)\right]^{\chi}}{\left[(-1)^{m}S\left(z_1^{-k}\dots z_{m}^{-k}\right)\right]^{\chi+\nu}} =$$

$$= \Bigg[\chi!\nu! \sum_{\alpha_1+\alpha_2+\dots+\alpha_r=\nu,\ \beta_1+\beta_2+\dots+\beta_p=\chi} \frac{1}{\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_p!} \times$$

$$\times \left(\frac{(z_1\cdots z_{m-\chi})^{\alpha_1}\cdots(z_{n-m+\chi+1}\cdots z_n)^{\alpha_r}(z_1\cdots z_{m+\nu})^{\beta_1}\cdots(z_{n-m-\nu+1}\cdots z_n)^{\beta_p}}{(z_1\cdots z_m)^{\chi+\nu}}\right)^{-k}\Bigg] \times$$

$$\times \left[(\chi+\nu)! \sum_{\lambda_1+\lambda_2+\dots+\lambda_q=\chi+\nu} \frac{1}{\lambda_1!\cdots\lambda_q!}\left(\frac{(z_1\cdots z_m)^{\lambda_1}\cdots(z_{n-m+1}\cdots z_n)^{\lambda_q}}{(z_1\cdots z_m)^{\chi+\nu}}\right)^{-k}\right]^{-1} = \frac{\varepsilon(k)}{1+\varepsilon_1(k)}, \tag{6}$$

where $r = C_n^{m-\varkappa};\ p = C_n^{m+\nu};\ q = C_n^m$; by the symbol S are denoted elementary symmetric functions of the reciprocals of the roots of the k-th transformed polynomial (3). The unit in the denominator is obtained when $\lambda_1 = \varkappa+\nu,\ \lambda_2 = \dots = \lambda_q = 0.$

The number of terms of which the numerator $\varepsilon(k)$ in (6) consists does not exceed $(C_n^{m-\varkappa})^{\nu}(C_n^{m+\nu})^{\varkappa}$ and does not depend on k. The number of terms $\varepsilon_1(k)$ is no greater than $(C_n^m)^{\varkappa+\nu}-1$, which also does not depend on k. The term of largest absolute value in $\varepsilon(k)$ will occur for the values $\alpha_1 = \nu,\ \alpha_2 = \dots = \alpha_r = 0,\ \beta_1 = \varkappa,\ \beta_2 = \dots = \beta_p = 0$:

$$\left|\frac{(z_1 \dots z_{m-\varkappa})^{-k\nu}(z_1 \dots z_{m+\nu})^{-k\varkappa}}{(z_1 \dots z_m)^{-k(\varkappa+\nu)}}\right| = \left|\frac{(z_{m-\varkappa-1} \dots z_m)^{k\nu}}{(z_{m+1} \dots z_{m+\nu})^{k\varkappa}}\right| \leqslant \left|\left(\frac{z_m}{z_{m+1}}\right)^{k\varkappa\nu}\right|.$$

An analogous assertion is valid for the moduli of each of the terms in $\varepsilon_1(k)$. Consequently, taking into account inequalities (2), (5), for sufficiently large values of k we may write

$$\left|\frac{(a_{m-\varkappa}^{(k)})^{\nu}(a_{m+\nu}^{(k)})^{\varkappa}}{(a_m^{(k)})^{\varkappa+\nu}}\right| = \frac{O\left(\left|\frac{z_m}{z_{m+1}}\right|^k\right)}{1+O_1\left(\left|\frac{z_m}{z_{m+1}}\right|^k\right)}.$$

Hence,

$$\lim_{k\to\infty}\frac{(a_{m-\varkappa}^{(k)})^{\nu}(a_{m+\nu}^{(k)})^{\varkappa}}{(a_m^{(k)})^{\varkappa+\nu}} = 0.$$

This implies (4).

Suppose now that, for the coefficient $a_m^{(k)}$, the equalities (4) hold. The maximum value of $\varkappa+\nu$ is not greater than n; therefore, for any pair of values $\varkappa$ and ν, $1 \le \varkappa \le m,\ 1 \le \nu \le n-m$, it follows that

$$\varkappa\nu \le \left(\frac{\varkappa+\nu}{2}\right)^2 \le \left(\frac{n}{2}\right)^2.$$

Put $\varepsilon = 9^{-n^2/4}$ and choose such an h that, for every pair of values $\varkappa, \nu$, the inequalities

$$\left|\frac{(a_{m-\varkappa}^{(k)})^{\nu}(a_{m+\nu}^{(k)})^{\varkappa}}{(a_m^{(k)})^{\varkappa+\nu}}\right| < 9^{-n^2/4},$$

or

$$\left|\frac{(a_{m-\varkappa}^{(k)})^{\nu}(a_{m+\nu}^{(k)})^{\varkappa}}{(a_m^{(k)})^{\varkappa+\nu}}\right|^{1/\varkappa\nu} = \frac{\left|\frac{a_{m-\varkappa}^{(k)}}{a_m^{(k)}}\right|^{1/\varkappa}}{\left|\frac{a_m^{(k)}}{a_{m+\nu}^{(k)}}\right|^{1/\nu}} < 9.$$

The last inequality is valid for every pair of values $\varkappa$ and ν, hence:

$$\frac{\min\limits_{1\le\nu\le n-m}\left|\dfrac{a_m^{(k)}}{a_{m+\nu}^{(k)}}\right|^{1/\nu}}{\max\limits_{1\le\varkappa\le m}\left|\dfrac{a_{m-\varkappa}^{(k)}}{a_m^{(k)}}\right|^{1/\varkappa}} > 9.$$

According to the results of Valiron ([4]), from the last inequality follows the validity of the inequality $|z_m^k| < |z_{m+1}^k|$, and hence also (5).

From the theorem we obtain

Corollary 1. If (5) is satisfied for the polynomial (1), then, beginning with some value h_0, the index m becomes the principal index of the Newton diagram $\mathfrak{M}_{f_k(z)}$ of the function $f_k(z)$ for $h < h_0$.

Corollary 2. If the transformation of the polynomials (3) is carried out by the method of "squaring the roots" ($k = 2^h$), then, for $|z_m| < |z_{m+1}|$, it follows from (4), in particular, that for sufficiently large k

$$|a_m^{(k)}|^2 \ge 2\,|a_{m-\nu}^{(k)} a_{m+\nu}^{(k)}|, \quad \text{where } \chi = \nu = 1, \dots, \min(m, n-m).$$

Example

$$f(z) = 1 - z + z^2 - z^3 + \dots + (-1)^{2n} z^{2n}.$$

Here $f_k(z) = 1 - z + z^2 - z^3 + \dots + (-1)^{2n} z^{2n}$, $k = 2^1, 2^2, \dots, 2^h$. For each coefficient $a_m^{(k)}$, for each pair of values χ and ν, the left-hand side of equality (4) is equal to one; consequently,

$$|z_1| = |z_2| = \dots = |z_n|,$$

although for the coefficients with even indices the generally accepted rule for determining "correctly" changing coefficients is satisfied, i.e.

$$a_m^{(k+1)} = (a_m^{(k)})^2.$$

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named after Ivan Franko

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Note: Figure translations are in progress. See original paper for figures.