MATHEMATICS

N. A. SAPOGOV

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

N. A. SAPOGOV

ON THE NORMS OF LINEAR POLYNOMIAL OPERATORS

(Presented by Academician V. I. Smirnov on 11 XII 1961)

1. Let $\widetilde{C}$ be the space of all complex-valued periodic continuous functions $f(x)$, $0 \le x \le 2\pi$, with norm

$$\|f\|_{\widetilde{C}} = \max_{0\le x\le 2\pi} |f(x)|.$$

By $U_n(f, x)$ we denote a linear operator defined in $\widetilde{C}$, mapping $\widetilde{C}$ into its subspace $\mathcal{E}_n$, formed by trigonometric polynomials

$$E_n(x) = \sum_{|k|\le n} c_k \exp(ikx)$$

of degree $\le n$, $n = 0, 1, 2, \ldots$ (c_k are complex numbers). Let, for any value $k = 0, \pm1, \pm2, \ldots$,

$$U_n(\exp(ikx), x) = \sum_{|h|\le n} \gamma^{(n)}_{k,h} \exp(ihx). \tag{1}$$

The conditions

$$\gamma^{(n)}_{k,k} = 1, \qquad \gamma^{(n)}_{k,h} = 0 \text{ for } k \ne h; \quad k, h = 0, \pm1, \ldots, \pm n, \tag{2}$$

are equivalent to the requirement that

A. $U_n(E_n, x) = E_n(x)$ for every polynomial $E_n(x) \in \widetilde{C}$.

2. Theorem 1. *If*

$$\sup_{\|E_n\|_{\widetilde{C}}\le 1} \|U_n(E_n, x) - E_n(x)\|_{\widetilde{C}} = \Delta_n, \tag{3}$$

where $\Delta_n < 1$ *and the supremum is taken over all* $E_n \in \widetilde{C}$, $\|E_n\| \le 1$, *then*

$$\|U_n\|_{\widetilde{C}} \ge \frac{4}{\pi^2}(1 - \Delta_n) \ln n + O(1). \tag{4}$$

Proof. For any $f \in \widetilde{C}$ we have:

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x-t)\,dt = \frac{1}{2\pi}\int_1^{2\pi} U_n(f_{nt}, x-t)\,dt, \tag{5}$$

where $f_t = f(x+t)$, $f_n = \sum_{|k|\leq n} c_k(f)\exp(ikx)$, $c_k f$ are the Fourier coefficients of the function f; t is regarded as a parameter. It is well known that

$$\sup_{\|f\|_{\widetilde{C}}\leq 1} \|f_n\|_{\widetilde{C}} = \frac{4}{\pi^2}\ln n + O(1). \tag{6}$$

Therefore, taking condition (3) into account, for any t we have

$$\|U_n(f_{nt}, x-t) - f_n(x)\|_{\widetilde{C}} \leq \Delta_n \frac{4}{\pi^2}\ln n + O(1).$$

Consequently,

$$\left\|\frac{1}{2\pi}\int_0^{2\pi} U_n(f_{nt}, x-t)\,dt - f_n(x)\right\|_{\widetilde{C}} \leq \Delta_n \frac{4}{\pi^2}\ln n + O(1).$$

Therefore, taking into account (5) and (6):

$$\left\|\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x-t)\,dt\right\|_{\widetilde{C}} \geq \frac{4}{\pi^2}\ln n - \Delta_n \frac{4}{\pi^2}\ln n + O(1),$$

whence inequality (4) follows. We note that in the case $\Delta_n = 1$ condition (3) is compatible with the equality $\|U_n\|_{\widetilde{C}} = 0$, which is realized for the operator $U_n(f, x) \equiv 0$ (for any f).

Corollary 1. There does not exist a sequence of linear operators $U_n(f, x)$, mapping $\widetilde{C}$ into $\mathcal{E}_n$, $n = 0, 1, 2, ...$, for which, for every $f \in \widetilde{C}$, the relations

$$\|U_n(f, x) - f(x)\|_{\widetilde{C}} \to 0, \quad n \to \infty, \quad \text{if } (1-\Delta_n)\ln n \to \infty$$

as $n \to \infty$, where the numbers $\Delta_n < 1$ are defined by equality (3), would hold.

Remark. The Lozinskii-Kharshiladze theorem (^{1,2}) is contained in Corollary 1 as a special case, when $\Delta_n = 0$ for all n. In this case condition (3) becomes condition A.

3. In the author' s note (^{3}) the identity

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x-t)\,dt = U_n^0(f,x); \tag{7}$$

was proved; here the linear operator U_n^0 is defined as follows: for any $f \in \widetilde{C}$

$$U_n^0(f,x) = \sum_{|k|\le n} c_k(f)\,\gamma_{k,k}^{(n)} \exp(ikx), \tag{8}$$

where $\gamma_{k,k}^{(n)}$ are the diagonal coefficients of the linear transformation (1). From identity (7) there follows the estimate for the norm of the operator U_n: $\|U_n\|_{\widetilde{C}} \ge \|U_n^0\|_{\widetilde{C}}$, whatever the coefficients $\gamma_{k,h}^{(n)}$, $|k| \le n$, $|h| \le n$, may be. This leads to the following theorem:

Theorem 2. If the diagonal coefficients $\gamma_{k,k}^{(n)}$ of the transformation (1) satisfy the condition

$$\delta_n^2 = \sum_{|k|\le n} |\gamma_{k,k}^{(n)} - 1|^2 < 1, \tag{9}$$

then

$$\|U_n\|_{\widetilde{C}} \ge \frac{4}{\pi^2}(1-\delta_n)\ln n + O(1). \tag{10}$$

Proof. For any polynomial $E_n(x) \in \mathcal{E}_n$ we have:

$$|U_n^0(E_n,x) - E_n(x)| \le \left|\sum_{|k|\le n} c_k(\gamma_{k,k}^{(n)} - 1)\exp(ikx)\right| \le$$

$$\le \left(\sum_{|k|\le n} |c_k|^2\right)^{1/2} \left(\sum_{|k|\le n} |\gamma_{k,k}^{(n)} - 1|^2\right)^{1/2} = \delta_n,$$

since $\sum_{|k|\le n} |c_k|^2 \le 1$, if $\|E_n\|_{\widetilde{C}} \le 1$. Therefore, by Theorem 1,

$$\|U_n^0\|_{\widetilde{C}} \ge \frac{4}{\pi^2}(1-\delta_n)\ln n + O(1),$$

which, together with the inequality $\|U_n\|_{\widetilde{C}} \ge \|U_n^0\|_{\widetilde{C}}$, proves Theorem 2.

Corollary 2. There does not exist a sequence of linear operators $U_n(f,x)$, mapping $\widetilde{C}$ into $\mathcal{E}_n$, $n = 0, 1, 2, \ldots$, for which fulfilled would hold for any $f \in \widetilde{C}$:

$$\|U_n(f,x) - f(x)\|_{\widetilde{C}} \to 0, \qquad n \to \infty,$$

if

$$\left[1 - \left(\sum_{|k|\le n} \left|\gamma_{k,k}^{(n)} - 1\right|^2\right)^{1/2}\right] \ln n \to \infty$$

as $p \to \infty$, where the numbers $\gamma_{k,k}^{(n)}$ are the diagonal coefficients of the transformations (1) corresponding to $n = 0, 1, 2, \ldots$.

4. Results analogous to those set forth above are also valid for algebraic polynomials. For example, the formulation of Theorem 1 carries over almost verbatim to this case.

Let C be the space of continuous (real-valued) functions $f(x)$ on the closed interval $[-1,1]$, $\mathcal{P}_n \subset C$, and let its elements be only polynomials

$$P_n(x) = \sum_0^n p_k x^k$$

of degree $\le n$; the norm is

$$\|f\|_C = \max_{|x|\le 1} |f(x)|.$$

By $U_n(f,x)$ we denote a linear operator defined on C and mapping this space into $\mathcal{P}_n$. Then the following theorem is valid:

Theorem 1a. If

$$\sup_{\|P_n\|_C \le 1} \|U_n(P_n,x) - P_n(x)\|_C = \Delta_n, \tag{11}$$

where $\Delta_n < 1$ and the supremum is taken over all $P_n \in C$, $\|P_n\| \le 1$, then

$$\|U_n\|_C \ge \frac{4}{\pi^2}(1 - \Delta_n)\ln n + O(1).$$

Corollary 1a. There does not exist a sequence of linear operators $U_n(f,x)$, mapping C into $\mathcal{P}_n$, $n = 0, 1, 2, \ldots$, for which the relations

$$\|U_n(f,x) - f(x)\|_C \to 0, \qquad n \to \infty$$

would hold (for every $f \in C$), if only

$$(1 - \Delta_n)\ln n \to \infty$$

as $n \to \infty$, where the numbers Δ_n are defined by equality (11).

Received
6 XII 1961

References

[1] S. M. Lozinskii, *DAN*, 61, No. 2 (1948).
[2] I. P. Natanson, *Constructive Theory of Functions*, 1949.
[3] N. A. Sapogov, *DAN*, 132, No. 1 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.