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## Abstract

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## EVOLUTE SURFACES OF A TWO-DIMENSIONAL DUALLY NORMALIZED $D_2$ IN $E_4$

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1. We shall say that a surface $X_m$ in the projective space $P_n$ is dually normalized if it is normalized in the sense of A. P. Norden ([1]) and its normal of the first kind contains the characteristic of the family of hyperplanes tangent to $X_m$. A number of properties of dually normalized surfaces were obtained in the author' s papers ([2]). In this note we shall restrict ourselves to the case of a surface of two dimensions $X_2$, immersed in the Euclidean $E_4$.

Suppose that $X_2$ can be completed to a hypersurface in such a way that the characteristics of the family of tangent hyperplanes are perpendicular to the tangent plane of the surface $X_2$. It is obvious that then the natural normalization of $X_2$ will at the same time also be dual. The surfaces $X_2$ admitting a dual normalization form a certain class, which in what follows we shall denote by $D_2$.

The basic differential equations of surfaces of the class $D_2$ have the form ([2])

$$\nabla_j r_i = h_{ij} X + k_{ij} Y, \qquad \mathbf{X}_j = -h_j^l r_l, \qquad \mathbf{Y}_j = -k_j^l r_l, \tag{A}$$

where $h_{ij} = -\partial_j r\, \partial_i X = \mathbf{X} \nabla_j r_i$, $k_{ij} = -\partial_j r\, \partial_i Y = \mathbf{Y} \nabla_j r_i$, and $\mathbf{X}$ and $\mathbf{Y}$ denote, respectively, the normal vector of the tangent hyperplane and the vector of the characteristic line.

2. Let us construct the congruence of normals of a dually normalized surface, going in the direction of the unit vectors

$$X^* = X \cos\alpha + Y \sin\alpha, \qquad \alpha = \text{const}.$$

$$\mathbf{R} = \mathbf{r}(u^1, u^2) + \rho X^*(u^1, u^2). \tag{1}$$

Let us pose the question: do focal surfaces of this congruence exist, i.e., such surfaces which are touched by all its rays? In this case there must exist a function $\rho(u^1, u^2)$ such that the vectors $\mathbf{R}_1, \mathbf{R}_2$ and $X^*$ are coplanar. Taking (1) and (A) into account, we obtain

$$\mathbf{R}_i = [\delta_i^s - \rho\,(h_i^s \cos\alpha + k_i^s \sin\alpha)]\,\mathbf{r}_s + \rho_{iX}^*.$$

Taking the latter into account, we must require the existence of numbers $\mu^1, \mu^2, \gamma$, not all simultaneously equal to zero, such that

$$\mu^i \mathbf{R}_i + \gamma X^* = 0,$$

i.e.

$$\mu^i\,[\delta_i^s - \rho\,(h_i^s \cos\alpha + k_i^s \sin\alpha)]\,\mathbf{r}_s + (\mu^i \rho_i + \gamma) X^* = 0.$$

Hence it follows that

$$\mu^i\,[\delta_i^s - \rho\,(h_i^s \cos\alpha + k_i^s \sin\alpha)] = 0.$$

Thus, the desired function $\rho(u^1, u^2)$ must be a quantity reciprocal to a root of the equation

$$\mathrm{Det}\,\|h_i^s \cos\alpha + k_i^s \sin\alpha - \omega\delta_i^s\| = 0. \tag{2}$$

Since the tensors $h_{ij}, k_{ij}$ have common principal directions ([26]), the roots of equation (2) are equal to

$$\omega_i = \sigma_i \cos\alpha + \tau_i \sin\alpha,$$

where $\sigma_i, \tau_i$ are the principal values of the tensors $h_{ij}, k_{ij}$, respectively.
Thus, on the ray (1) there exist two desired focal points

$$\underset{1}{\mathbf{R}} = \mathbf{r} + \frac{1}{\sigma_1 \cos\alpha + \tau_1 \sin\alpha}(\mathbf{X}\cos\alpha + \mathbf{Y}\sin\alpha),$$

$$\underset{2}{\mathbf{R}} = \mathbf{r} + \frac{1}{\sigma_2 \cos\alpha + \tau_2 \sin\alpha}(\mathbf{X}\cos\alpha + \mathbf{Y}\sin\alpha). \tag{3}$$

If in (1) we put $\alpha = 0$ and $\alpha = \pi/2$, then we obtain the congruences of normals $D_2$ going in the directions of the unit vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively.
Thus, on each normal to $D_2$ there exist two focal points. Let us determine how the focal points are situated in the normal plane to $D_2$. Introduce in this plane a rectangular coordinate system with origin at the point of the surface $D_2$ and with axes going in the directions of the unit vectors $\mathbf{X}, \mathbf{Y}$. Then from (3) it follows that the coordinates of the focal points $F_1(\underset{1}{\mathbf{R}})$ are equal to

$$x = \frac{\cos\alpha}{\sigma_1\cos\alpha + \tau_1\sin\alpha}, \qquad y = \frac{\sin\alpha}{\sigma_1\cos\alpha + \tau_1\sin\alpha}.$$

Eliminating the parameter $\alpha$, we obtain the equation

$$\sigma_1 x + \tau_1 y = 1. \tag{4}$$

Thus, the focal points $F_1(\underset{1}{\mathbf{R}})$ are situated in the normal plane on a straight line not passing through the point of the surface $D_2$. Similarly, we obtain that the focal points $F_2(\underset{2}{\mathbf{R}})$ lie on the straight line

$$\sigma_2 x + \tau_2 y = 1. \tag{5}$$

We shall call the straight lines (4) and (5) the **axes of curvature** of the surface $D_2$. From (4) and (5) we find the geometric meaning of the invariants $\chi_1 = \sqrt{\sigma_1^2 + \tau_1^2}$ and $\chi_2 = \sqrt{\sigma_2^2 + \tau_2^2}$: their reciprocals $\frac{1}{\chi_1}$ and $\frac{1}{\chi_2}$ give the distances from the point of the surface $D_2$ to the axes of curvature (4) and (5), respectively.

Let us pose the question: do there exist envelopes of the family of normal planes of the surface $D_2$?
Consider the points

$$\mathbf{R} = \mathbf{r}(u^1, u^2) + a(u^1, u^2)\mathbf{X} + b(u^1, u^2)\mathbf{Y}$$

in the normal plane $D_2$, and require that at these points it be tangent to some surface, i.e. that the derivatives $\mathbf{R}_1, \mathbf{R}_2$ decompose with respect to $\mathbf{X}$ and $\mathbf{Y}$. We have

$$\mathbf{R}_i = (\delta_i^s - a h_i^s - b k_i^s)\mathbf{r}_s + a_i\mathbf{X} + b_i\mathbf{Y},$$

whence we obtain the condition

$$\delta_i^s - a h_i^s - b k_i^s = 0. \tag{6}$$

Let $a^i$ and $\tilde{a}^i$ be common principal directions of the tensors $h_{ij}$ and $k_{ij}$. Then from (6) we have

$$a^i - a\sigma_1 a^i - b\tau_1 a^i = 0, \qquad \tilde{a}^i - a\sigma_2\tilde{a}^i - b\tau_2\tilde{a}^i = 0$$

or

$$a\sigma_1 + b\tau_1 = 1, \quad a\sigma_2 + b\tau_2 = 1. \tag{7}$$

In this case our requirements are satisfied and the envelope exists. Comparing (7) with (4) and (5), we see that $a(u^1, u^2)$ and $b(u^1, u^2)$ are the coordinates of the points of intersection of the axes of curvature, which, thus, lie on the envelope of the family of normal planes $D_2$. Thus, we have proved that the normal planes $D_2$, embedded in $E_4$, admit an enveloping surface. We shall call this surface the evolute surface.

We shall now prove that if the normal planes of a two-dimensional surface embedded in $E_4$ admit an envelope, then this surface is dually normalizable.

Let

$$\mathbf{R} = \mathbf{R}(u^1, u^2)$$

be the parametric equation of the evolute surface. If $\mathbf{m}, \mathbf{n}$ are unit and mutually perpendicular normal vectors of the evolute surface, then its basic derivative equations have the form

$$\nabla_j \mathbf{R}_i = a_{ij}\mathbf{m} + b_{ij}\mathbf{n}, \tag{8}$$

where $a_{ij} = -\partial_j \mathbf{R}\, \partial_i \mathbf{m} = \mathbf{m}\nabla_j \mathbf{R}_i$, $b_{ij} = -\partial_j \mathbf{R}\, \partial_i \mathbf{n} = \mathbf{n}\nabla_j \mathbf{R}_i$, and $\nabla$ is the symbol of covariant differentiation in the intrinsic connection of the evolute surface. If the radius vector of a point of the original surface is represented in the form

$$\mathbf{R}^* = \mathbf{R}(u^1, u^2) + \lambda^k(u^1, u^2)\mathbf{R}_k, \tag{9}$$

then we must require that the derivatives $\mathbf{R}^*_1, \mathbf{R}^*_2$ be resolved along $\mathbf{m}$ and $\mathbf{n}$. We have:

$$\mathbf{R}^*_i = (\delta^m_i + \nabla_i \lambda^k)\mathbf{R}_k + \lambda^k a_{ik}\mathbf{m} + \lambda^k b_{ik}\mathbf{n},$$

whence we obtain the condition

$$g_{ij} + \nabla_i \lambda_j = 0; \tag{10}$$

$\lambda_i$ is a gradient, since $g_{ij}$ is symmetric.

Forming the integrability condition for (10), we obtain $R_{\cdot kij}{}^{l}\lambda_l = 0$, or [(3)]

$$K\varepsilon_{jk}\varepsilon_i{}^l\lambda_l = 0, \tag{11}$$

where $K$ is the Gaussian curvature of the intrinsic geometry of the evolute surface. From (11) it follows that $K = 0$, and this means that the intrinsic geometry of the evolute surface is Euclidean. The line element of this surface can be reduced to the form $ds^2 = du^2 + dv^2$. Hence, in turn, it follows that

$$\mathbf{R}_1^2 = 1, \qquad \mathbf{R}_2^2 = 1, \qquad \mathbf{R}_1\mathbf{R}_2 = 0.$$

From (10) we obtain

$$\partial_1\lambda_1 = -1, \qquad \partial_2\lambda_2 = -1, \qquad \partial_1\lambda_2 = \partial_2\lambda_1 = 0;$$

whence it follows that

$$\lambda_1 = -u + a, \qquad \lambda_2 = -v + b,$$

where $a, b$ are constants.

From what has been said it is clear that equation (9) can be rewritten in the form

$$\mathbf{R}^* = \mathbf{R}(u, v) + (a - u)\mathbf{R}_1 + (b - v)\mathbf{R}_2. \tag{12}$$

It is easy to see that equation (12) represents the surface $D_2$. Indeed, the vectors $\mathbf{R}_1, \mathbf{R}_2$ are unit vectors, and for them the conditions $\partial_i\mathbf{R}_j = a_{ij}\mathbf{m} + b_{ij}\mathbf{n}$ hold, where $\mathbf{m}$ and $\mathbf{n}$ are normal vectors. Hence

$$\mathbf{R}_1\partial_j\mathbf{R}_2 = 0, \qquad \mathbf{R}_2\partial_j\mathbf{R}_1 = 0, \qquad j = 1, 2.$$

This means that the surface is doubly normalized.

Thus, we have proved the following theorems:

**Theorem 1.** *In order that a surface be doubly normalizable, it is necessary and sufficient that its normal planes have an envelope of zero curvature.*

**Theorem 2.** *Every $D_2$ in $E_4$ cuts orthogonally the tangent planes of some surface of zero curvature.*

Thus, the problem of finding $D_2$ in $E_4$ is equivalent to the problem of finding $X_2$ of zero curvature. We shall call surfaces that cut orthogonally the tangent planes of a given surface its **evolvent** surfaces. Then this result can be formulated as follows:

*Every $D_2$ in $E_4$ is an evolvent surface of an arbitrary surface of zero curvature.*

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3. A. P. Norden, *Theory of Surfaces*, 1956.

*Note: Figure translations are in progress. See original paper for figures.*