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1962

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Abstract

Full Text

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ON SOME REGULARLY MONOTONE POLYNOMIALS LEAST DEVIATING FROM ZERO

(Presented by Academician S. N. Bernstein on 24 II 1962)

Let λ be a positive integer. Denote, as was done in ([2]), by $_{\lambda,m}$ the class of regularly monotone functions of order m on $[0,1]$ for which the first and last type numbers do not exceed λ, while all the remaining type numbers are equal to λ. The class $_{\lambda,m}$ can be divided into 2λ nonintersecting subclasses $^{(j)}_{\lambda,m}$ in the following way: a function $f(x) \in {}^{(j)}_{\lambda,m}$ $(j = 1, \dots, \lambda)$, if $f(x) \in {}_{\lambda,m}$, $f(x)f'(x) \geqslant 0$ for $x \in [0,1]$, and the first type number of this function is equal to $\lambda + 1 - j$; a function $f(x) \in {}^{(j)}_{\lambda,m}$ $(j = \lambda+1, \dots, 2\lambda)$, if $f(x) \in {}_{\lambda,m}$, $f(x)f'(x) \leqslant 0$ for $x \in [0,1]$, and the first type number of this function is equal to $2\lambda + 1 - j$.

In ([2]) there were also introduced into consideration the generalized Euler-Bernstein polynomials $A_{j,m}(x) \in {}^{(j)}_{\lambda,m}$, having degree m and satisfying the conditions:

$$A^{(m)}_{j,m}(x) = 1, \qquad A^{(k)}_{j,m}(\alpha^{(j)}_k) = 0 \quad (k = 0, 1, \dots, m-1),$$

in which

$$\alpha^{(j)}_k = \begin{cases} 0 & \text{for } k = 0, 1, \dots, \lambda - j, \\ 1 & \text{for } k \equiv \lambda - j + 1, \dots, 2\lambda - j \pmod{2\lambda}, \\ 0 & \text{for } k \equiv 2\lambda - j + 1, \dots, 3\lambda - j \pmod{2\lambda}, \end{cases}$$

if $1 \leqslant j \leqslant \lambda$, and

$$\alpha^{(j)}_k = \begin{cases} 1 & \text{for } k = 0, 1, \dots, 2\lambda - j, \\ 0 & \text{for } k \equiv 2\lambda - j + 1, \dots, 3\lambda - j \pmod{2\lambda}, \\ 1 & \text{for } k \equiv 3\lambda - j + 1, \dots, 4\lambda - j \pmod{2\lambda}, \end{cases}$$

if $\lambda + 1 \leqslant j \leqslant 2\lambda$.

The introduction of 2λ sequences of generalized Euler-Bernstein numbers $E^{(j)}_m$ $(j = 1, \dots, 2\lambda;\ m = 0, 1, 2, \dots)$, defined for each fixed j by the recurrence relations

$$E^{(j)}_0 = 1,$$

$$(1+E^{(j)})_m = 0 \quad \text{for } m \equiv j, \dots, j+\lambda-1 \pmod{2\lambda},$$

$$E_m^{(j)} = 0 \quad \text{for } m \equiv j+\lambda, \dots, j+2\lambda-1 \pmod{2\lambda},$$

allows one to write any polynomial $A_{j,m}(x)$ in the form*

$$A_{j,m}(x) = \frac{(x+E^{(m+j)})_m}{m!},$$

* The symbol $(x+E)_m$ means that the parentheses are to be expanded by Newton' s binomial formula, and then the powers E^k are to be replaced by the numbers E_k.

where the index $m+j$ must be replaced by $m+j-2\lambda$, if $m+j > 2\lambda$. From a theorem of S. N. Bernstein ([1], p. 515) there follows the following extremal theorem:

Theorem 1. *Of all polynomials $P_m(x) \in {}^{(j)}_{\lambda,m}$ of the form*

$$P_m(x) = \frac{1}{m!}x^m + p_{m-1}x^{m-1} + \dots + p_0$$

the polynomial $A_{j,m}(x)$ deviates least from zero at each point $x \in [0,1]$. The magnitude of the least deviation on the whole interval $L_m^{(j)}$ is determined by the equalities

$$L_m^{(j)} = \left|A_{j,m}\left(1-\alpha_0^{(j)}\right)\right| = \begin{cases} \dfrac{\left|E_m^{(m+j+\lambda)}\right|}{m!}, & \text{for } j = 1, \dots, \lambda, \\ \dfrac{\left|E_m^{(m+j)}\right|}{m!}, & \text{for } j = \lambda+1, \dots, 2\lambda. \end{cases}$$

The following propositions hold, making it possible to compare the quantities $L_m^{(j)}$ for different j.

Theorem 2. *If $1 \le j \le \lambda$, then $L_m^{(j)} = L_m^{(\lambda+j)}$.*

Theorem 3. *If $1 \le j \le \lambda$, then $L_{\lambda p+j}^{(\lambda+1-j)} = L_{\lambda p+j}^{(\lambda)}$.*

Theorem 4. *Of all polynomials $P_{\lambda p+j}(x) \in {}_{\lambda,\lambda p+j}$ of the form*

$$P_{\lambda p+j}(x) = \frac{1}{(\lambda p+j)!}x^{\lambda p+j} + p_{\lambda p+j-1}x^{\lambda p+j-1} + \dots + p_0$$

the polynomials $A_{\lambda+1-j,\lambda p+j}(x)$, $A_{2\lambda+1-j,\lambda p+j}(x)$, $A_{\lambda,\lambda p+j}(x)$, *and* $A_{2\lambda,\lambda p+j}(x)$ *deviate least from zero on* $[0,1]$, *and the magnitude of the least deviation is determined by the formula*

$$L_{\lambda p+j} = \frac{\left|E_{\lambda p+j}^{(\lambda p+1)}\right|}{(\lambda p+j)!} = \frac{\left|E_{\lambda p+j}^{(\lambda p+j)}\right|}{(\lambda p+j)!}.$$

Received
7 II 1962

CITED LITERATURE

[1] S. N. Bernstein, *On Certain Properties of Cyclically Monotone Functions*, Collected Works, 2, Publishing House of the Academy of Sciences of the USSR, 1954, p. 493.
[2] V. L. Fainshmidt, *Izv. AN BSSR*, Series of Physical-Technical Sciences, No. 3 (1960).

Note: Figure translations are in progress. See original paper for figures.