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MATHEMATICS

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1962

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Abstract

Full Text

MATHEMATICS

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ON A NEW METHOD FOR SOLVING THE BOUNDARY-VALUE PROBLEM FOR THE LINEARIZED SYSTEM OF NAVIER-STOKES EQUATIONS IN THE CASE OF THE PLANE

(Presented by Academician I. M. Vinogradov, 29 III 1962)

The linearized system of Navier–Stokes equations in the case of a stationary flow can be written in the form

$$\Delta \mathbf{u} - \text{grad } p = 0, \quad \text{div } \mathbf{u} = 0. \quad (1)$$

We assume that the external forces have a potential; therefore \mathbf{u} is the velocity vector, and p is a scalar function equal to the sum of the pressure and the potential of the external forces divided by the viscosity coefficient. Subsequently it will be shown how the inhomogeneous system (1) can be reduced to a homogeneous one.

We shall seek a solution of system (1) in the disk D ($|z| \leq 1$), when on its boundary Γ the components of the velocity are prescribed,

$$\mathbf{u}|_{\Gamma} = (\varphi_1, \varphi_2). \quad (2)$$

Introducing the complex variables $z = x + iy$, $w = u + iv$ and the derivatives

$$\begin{aligned} \frac{\partial w}{\partial \bar{z}} &= \frac{1}{2} \left\{ \left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y} \right) + i \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \right\}; \\ \frac{\partial w}{\partial z} &= \frac{1}{2} \left\{ \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) + i \left(-\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \right\}, \end{aligned}$$

we rewrite our system (1) in the form

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} - \frac{1}{2} \frac{\partial p}{\partial \bar{z}} = 0, \quad \text{Re } \frac{\partial w}{\partial z} = 0. \quad (3)$$

In view of the fact that $p(x, y)$ is a real function, system (3) can be integrated with respect to $w(z)$:

$$w(z) = \frac{1}{2} \int_0^z \Phi(t) dt - \frac{z}{2} \overline{\Phi(z)} + \Psi(z), \quad (4)$$

where $\Phi(z)$ and $\Psi(z)$ are arbitrary holomorphic functions in the disk D . Thus we have obtained a linear boundary-value problem for two holomorphic functions $\Phi(z)$ and $\Psi(z)$, since the values of $w(z)$ are prescribed on Γ by (2).

We multiply equation (4) by $\frac{1}{2\pi i} \frac{\overline{dt}}{t-z}$, integrate over Γ , and then subtract the equation thus obtained from (4); then we obtain the following relation:

$$w(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{w(t)}{t-z} \overline{dt} = \frac{1}{2} \int_0^z \Phi(t) dt - \frac{|z|^2 - 1}{2z} \overline{\Phi(z)} - \frac{1}{2z} \overline{\Phi(0)}. \quad (5)$$

Now let the point z tend to some boundary point $z_0 \in \Gamma$, remaining inside D . Using the formulas for the limiting values of an integral of Cauchy type and the relation $z\bar{z} = 1$ on the boundary Γ , we obtain on Γ

$$\frac{w(z_0)}{2} + \frac{1}{2\pi i} \int_{\Gamma} \frac{w(t)}{t-z_0} \overline{dt} = \frac{1}{2} \int_0^{z_0} \Phi(t) dt - \frac{z_0}{2} \overline{\Phi(0)}. \quad (6)$$

On the right-hand side of (6) stands the boundary value of a function holomorphic inside the unit disk; on the left-hand side stands the boundary value from within of the integral

$$-\frac{z}{2\pi i} \int_{\Gamma} \frac{w(t)}{1-zt} \overline{dt},$$

which is a function holomorphic inside D . Thus, continuing (6) into D , we obtain the equation

$$-\frac{z}{2\pi i} \int_{\Gamma} \frac{w(t)}{1-zt} \overline{dt} = \frac{1}{2} \int_0^z \Phi(t) dt - \frac{z}{2} \overline{\Phi(0)}. \quad (7)$$

In order that the functional equation (7) be solvable with respect to $\Phi(z)$, it is necessary and sufficient that the condition

$$\operatorname{Re} \left\{ \frac{1}{2\pi i} \int_{\Gamma} w(t) \overline{dt} \right\} = 0 \quad (8)$$

be satisfied.

If we denote

$$a = -\frac{1}{2\pi i} \int_{\Gamma} w(t) \bar{dt},$$

then from (7)

$$\frac{1}{2} \Phi(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{w(t)}{(1-zt)^2} \bar{dt} - \frac{a}{2}. \quad (9)$$

Substituting (9) into (5), we obtain the solution of problem (1)–(2)

$$\begin{aligned} w(z) = & -\frac{1}{2\pi i} \int_{\Gamma} \frac{w(t)}{t-z} \bar{dt} - \frac{z}{2\pi i} \int_{\Gamma} \frac{w(t)}{1-zt} \bar{dt} - \\ & - \frac{|z|^2 - 1}{2z\pi i} \int_{\Gamma} \frac{w(t)}{(1-zt)^2} \bar{dt} - az + \frac{a}{z}. \end{aligned} \quad (10)$$

Condition (8) has a simple hydromechanical meaning: it expresses that the inflow of liquid through the boundary contour Γ is equal to zero.

Let us now clarify the smoothness of the solution of our problem. From formula (12) it is seen that, since the solution is represented by an integral of Cauchy type and its derivative, the solution is an analytic function of z and \bar{z} . Further, if one assumes that the value $w(t)$ on Γ satisfies a Hölder condition with exponent ν , then the derivatives with respect to z and \bar{z} of $w(z)$, as it approaches the boundary, will grow no faster than $(1-|z|)^{\nu-1}$. Therefore, in order that $w(z)$ have derivatives summable with power $p \geq 1$, it is sufficient that

$$\nu > \frac{p-1}{p}.$$

Let us now consider the nonhomogeneous system (1) in complex notation

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} - \frac{1}{2} \frac{\partial p}{\partial z} = f(z), \quad \operatorname{Re} \frac{\partial w}{\partial z} = \varphi(z). \quad (11)$$

In order to reduce it to the homogeneous one, set

$$w(z) = w_1(z) - \frac{1}{\pi} \iint_D \frac{\varphi(t) + i\psi(t)}{t - \bar{z}} d\sigma_t, \quad (12)$$

where $\psi(t)$ is some real function.

Then system (11) is written in the form

$$\frac{\partial^2 w_1}{\partial z \partial \bar{z}} - 2 \frac{\partial p}{\partial z} + \frac{\partial(\varphi + i\psi)}{\partial \bar{z}} = f(z), \quad \operatorname{Re} \frac{\partial w_1}{\partial z} = 0.$$

Next we choose the real function $\psi(z)$ so that the expression

$$i \partial \psi / \partial \bar{z} - f(z)$$

is the derivative $\partial m / \partial z$ of some real function $m(z)$. Obviously,

$$\psi(z) = -\frac{1}{2\pi i} \iint_D \frac{f(t)}{t-z} d\sigma_t + \frac{1}{2\pi i} \iint_D \frac{\overline{f(t)}}{t-\bar{z}} d\sigma_t, \quad (13)$$

and system (II) becomes homogeneous

$$\frac{\partial^2 w_1}{\partial z \partial \bar{z}} - 2 \frac{\partial p_1}{\partial z} = 0, \quad \operatorname{Re} \frac{\partial w_1}{\partial \bar{z}} = 0, \quad (14)$$

where

$$p_1(z) = p(z) - \frac{1}{2} \varphi(z) + \frac{1}{2} \operatorname{Re} \iint_D \frac{f(t)}{t-z} d\sigma_t. \quad (15)$$

Passing from the complex notation to the real one, we obtain a reduction of the nonhomogeneous system (1) to a homogeneous one.

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Received
22 III 1962

REFERENCES

1. I. N. Vekua, *Generalized Analytic Functions*, 1959.

Note: Figure translations are in progress. See original paper for figures.

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