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Abstract

Full Text

MATHEMATICS

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ON THE CONVERGENCE OF VARIATIONAL PROCESSES

(Presented by Academician S. L. Sobolev on 20 IV 1962)

In the work [1], V. P. Il' in studied in detail the question of the uniform convergence of a minimizing sequence to the solution of a variational problem, depending on the rate of its convergence in one or another integral metric. With respect to the domain in which a certain class of functions is considered, V. I. Il' in assumed that it satisfies the "cone condition."

In the present work, on the basis of results obtained earlier by us [2], it is shown how the order of convergence in the integral metric changes depending on the magnitude of the zero angle of the domain.

We shall call a closed n -dimensional domain V_n , bounded by surfaces whose equations in a given rectangular coordinate system have the form

$$x_2^2 + \dots + x_n^2 = \alpha_0^2 x_1^{2\lambda}, \quad x_1 = a \quad (x_1 \geq 0, \lambda \geq 1, \alpha_0 > 0),$$

a conical body with parameters a, α_0, λ .

By $W_p^{(l)}(\Omega)$ we shall denote the set of functions summable in the n -dimensional domain Ω , obtained by closing the set of $C^{(l)}$ functions having continuous partial derivatives in Ω up to order l inclusive, with respect to the norm

$$\|f\|_{W_p^{(l)}(\Omega)} = \|f\|_{L_p(\Omega)} + \| |D^l f| \|_{L_p(\Omega)},$$

where

$$|D^l f| = \left(\sum_{i_1 \dots i_l=1}^n \left| \frac{\partial^l f}{\partial x_{i_1} \dots \partial x_{i_l}} \right|^2 \right)^{1/2}.$$

With respect to the domain Ω , it is assumed that each point of its boundary is reached by a body congruent to the fixed conical body V_n and lying entirely in $\bar{\Omega}$.

Theorem 1. Suppose that the function $f \in W_p^{(l)}(\Omega)$ and, moreover, the following conditions hold:

- a) $lp \leq \lambda(n-1) + 1$;
- b) all derivatives of order k are summable with exponent $\nu > 1$ on each section $\Omega_t : x_{t+1} = \text{const}, \dots, x_n = \text{const}$ ($t \leq n$), and the inequality

$$\| |D^k f| \|_{L_\nu(\Omega_t)} \leq R$$

holds for any choice of Ω_t ;

- c) $k\nu > \lambda(t-1) + 1$.

Then at each point of the domain $\bar{\Omega}$ the following estimates hold:

$$|f(P)| \leq C'_1 \|f\|_{L_p(\Omega)} + C''_1 \left(\| |D^l f| \|_{L_p(\Omega)}^{k \frac{\lambda(t-1)+1}{\nu}} R^{\frac{\lambda(n-1)+1}{p} - l} \right)^{1/\chi}, \quad (1)$$

if $\| |D^l f| \|_{L_p(\Omega)} \leq a^X R$, $lp < \lambda(n-1) + 1$;

$$|f(P)| \leq C'_2 \|f\|_{L_p(\Omega)} + C_2 \| |D^l f| \|_{L_p(\Omega)}, \quad (2)$$

if

$$\| |D^l f| \|_{L_p(\Omega)} > a^X R, \quad lp < \lambda(n-1) + 1,$$

or

$$|f(P)| \leq C'_3 \|f\|_{L_p(\Omega)} + C''_3 \| |D^l f| \|_{L_p(\Omega)} + C'''_3 \| |D^l f| \|_{L_p(\Omega)} \left| \ln \frac{R}{\| |D^l f| \|_{L_p(\Omega)}} \right|^{\frac{1}{p'}}, \quad (3)$$

if

$$\| |D^l f| \|_{L_p(\Omega)} \leq a^X R, \quad lp = \lambda(n-1) + 1 \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right);$$

$$|f(P)| \leq C'_4 \|f\|_{L_p(\Omega)} + C''_4 \| |D^l f| \|_{L_p(\Omega)}, \quad (4)$$

if

$$\| |D^l f| \|_{L_p(\Omega)} > a^X R, \quad lp = \lambda(n-1) + 1,$$

where C with various indices are constants independent of f ; P is an arbitrary point of the domain Ω ;

$$\chi = \frac{\lambda(n-1)+1}{p} - l + k - \frac{\lambda(t-1)+1}{v}.$$

Corollary. From inequalities (1) and (2) the following estimates follow immediately:

$$|f(P)| \leq C_5 \left(\|f\|_{W_p^{(l)}(\Omega)}^{k - \frac{\lambda(t-1)+1}{v}} R^{\frac{\lambda(n-1)+1}{p} - l} \right)^{\frac{1}{\chi}} \quad \text{when } \|f\|_{W_p^{(l)}(\Omega)} \leq a^\chi R; \quad (5)$$

$$|f(P)| \leq C_6 \|f\|_{W_p^{(l)}(\Omega)} \quad \text{when } \|f\|_{W_p^{(l)}(\Omega)} > a^\chi R; \quad (6)$$

C_5, C_6 are constants independent of the function f .

Theorem 2. Under the assumptions of Theorem 1, the following estimate holds:

$$|f(P)| \leq C_7 \left(\|f\|_{L_p(\Omega)}^{k - \frac{\lambda(t-1)+1}{v}} R^{\frac{\lambda(n-1)+1}{p}} \right)^{\frac{1}{\chi_1}}, \quad (7)$$

where $\|f\|_{L_p(\Omega)} \leq a^{\chi_1} R$,

$$\chi_1 = k - \frac{\lambda(t-1)+1}{v} + \frac{\lambda(n-1)+1}{p}.$$

The constant C_7 does not depend on the function f .

Let us give an example showing that estimate (5), obtained above, is exact in order.

Example. Consider the function defined on the interval $[0, \infty)$:

$$\psi(t) = \begin{cases} (1-t)^{l+1}(1+a_1t+\dots+a_{l-1}t^{l-1}) = 1+b_{1t}^l+\dots+b_{2t}^{2l}, & \text{for } 0 \leq t \leq 1, \\ 0, & \text{for } t > 1, \end{cases}$$

and the sequence of functions $u_i(x_1, \dots, x_n) = i^\beta \psi(ix_1)$, defined on the conic body

$$V_n \{x_2^2 + \dots + x_n^2 = a_0^2 x_1^{2\lambda}, \quad 0 \leq x_1 \leq 1, \quad \lambda \geq 1, \quad a_0 > 0\};$$

$\beta > 1$ is a real number.

Elementary calculations show that the order of convergence of the right-hand side of inequality (5) for the sequence $\{u_i\}$ coincides with the order of its uniform convergence.

Theorem 3. Let $u \in W_p^{(l)}(\Omega)$, $lp \leq \lambda(n-1) + 1$; let $\{u_i\} \in W_p^{(l)}(\Omega)$ be a sequence of functions of the form

$$u_i(x_1, \dots, x_n) = \sum_{j_1 \dots j_n=0}^i a_{j_1 \dots j_n} x_1^{j_1} \dots x_n^{j_n}. \quad (8)$$

Suppose, further, that it is given that E_i decrease monotonically and

$$\lim E_i i^{2[\frac{\lambda(n-1)+1}{p}-l]} (\ln i)^\alpha = 0, \quad lp < \lambda(n-1) + 1;$$

$$\lim E_i \left(\ln \frac{i}{E_i} \right)^{\frac{1}{p'}} (\ln i)^\alpha = 0, \quad lp = \lambda(n-1) + 1;$$

$$\alpha > 1, \quad E_i = \|u - u_i\|_{W_p^{(l)}(\Omega)} \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right).$$

Then $u(x_1, \dots, x_n)$ is equivalent in $\bar{\Omega}$ to a continuous function $\bar{u}(x_1, \dots, x_n)$, to which the sequence $\{u_i\}$ converges uniformly, and moreover

$$|\bar{u} - u_i| \leq C_8 E_i i^{2[\frac{\lambda(n-1)+1}{p}-l]} + C_8 \sum_{s=r}^{\infty} E_{2^s} (2^s)^{2[\frac{\lambda(n-1)+1}{p}-l]}, \quad lp < \lambda(n-1) + 1;$$

$$|\bar{u} - u_i| \leq C_9 E_i \left(\ln \frac{i}{E_i} \right)^{\frac{1}{p'}} + C_9 \sum_{s=r}^{\infty} E_{2^s} \left(\ln \frac{2^s}{E_{2^s}} \right)^{\frac{1}{p'}}, \quad lp = \lambda(n-1) + 1.$$

The constants C_8, C_9 do not depend on the functions \bar{u}, u_i ; $2^{r-1} < i \leq 2^r$.

If in the hypotheses of Theorem 3 we put $l = 1, p = 2$, then we obtain sufficiently precise conditions for uniform convergence of the minimizing sequence $\{u_i\}$ to the solution of the variational problem corresponding to the problem of solving an elliptic equation of second order under homogeneous conditions (see (3)).

For $p = 2, l = m$, an analogous result is obtained for a polyharmonic equation of order $2m$.

Theorem 4. Let $u \in W_2^{(1)}(\Omega)$, and let the functions u_i be polynomials of the form (8). Suppose further that

$$E_i = \|u_i - u\|_{W_2^{(1)}(\Omega)};$$

then, if

$$\lim_{i \rightarrow \infty} E_i i^{\lambda(n-1)+1} (\ln i)^\alpha = 0 \quad (\alpha > 1),$$

the sequence of derivatives $\{\partial u_i / \partial x_j\}_{j=1,2,\dots}$ converges uniformly in $\bar{\Omega}$ to $\partial u / \partial x_j$, and moreover

$$\left| \frac{\partial u}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \right| \leq C_{10} E_i i^{\lambda(n-1)+1} + C_{10} \sum_{s=m}^{\infty} E_{2^s} 2^{s[\lambda(n-1)+1]},$$

where $2^{m-1} < i \leq 2^m$, C_{10} does not depend on u, u_i .

Theorem 5. Let $u \in W_2^{(2)}(\Omega)$, and let the functions u_i be polynomials of the form (8). Suppose further that

$$E_i = \|D^2(u - u_i)\|_{L_2(\Omega)} + \|D(u - u_i)\|_{L_2(\Omega)} + \|u - u_i\|_{L_2(\Omega)}.$$

Then, if the conditions

$$\lim_{i \rightarrow \infty} E_i i^{\lambda(n-1)-1} (\ln i)^\alpha = 0 \quad (\alpha > 1) \quad \text{for } \lambda(n-1) > 1,$$

$$\lim_{i \rightarrow \infty} E_i \left(\ln \frac{i}{E_i} \right)^{1/2} (\ln i)^\alpha = 0 \quad (\alpha > 1) \quad \text{for } \lambda(n-1) = 1,$$

are satisfied, then $\{\partial u_i / \partial x_j\}$ converges uniformly to $\partial u / \partial x_j$, and moreover

$$\left| \frac{\partial u}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \right| \leq C_{11} E_i i^{\lambda(n-1)-1} + C_{11} \sum_{s=m}^{\infty} E_{2^s} 2^{s[\lambda(n-1)-1]} \quad \text{for } \lambda(n-1) > 1;$$

$$\left| \frac{\partial u}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \right| \leq C_{12} E_i \left(\ln \frac{i}{E_i} \right)^{1/2} + C_{12} \sum_{s=m}^{\infty} E_{2^s} \left(\ln \frac{2^s}{E_{2^s}} \right)^{1/2} \quad \text{for } \lambda(n-1) = 1,$$

where $2^{m-1} < i \leq 2^m$; C_{11}, C_{12} do not depend on u, u_i .

An analogous theorem can also be formulated for the uniform convergence of second derivatives.

In the theorems presented, the functions u_i were assumed to be polynomials in x_1, \dots, x_n ; however, all the arguments in our theorems remain valid if, as functions of x_1, \dots, x_n , they are of some other definite type—what is important is only that estimates of the type of A. A. Markov's theorem hold for them (see ⁽⁴⁾). For example, if the u_i are trigonometric polynomials, then all the theorems proved remain valid, since for them an analogous theorem of S. N. Bernstein is valid (see ⁽⁴⁾).

In ⁽⁵⁾ it is shown that if the functions $u_i(x_1, \dots, x_n)$ are functions of the form

$$u_i(x_1, \dots, x_n) = \omega(x_1, \dots, x_n) \sum_{j_1 \dots j_n=0}^i a_{j_1 \dots j_n} x_1^{j_1} \dots x_n^{j_n} = \omega P_i, \quad (9)$$

where $\omega(x_1, \dots, x_n)$ is a twice continuously differentiable function defined in an open domain containing $\bar{\Omega}$, satisfying the conditions:

- 1) $\omega(x_1, \dots, x_n)|_S = 0$;
- 2) $\omega(x_1, \dots, x_n) > 0$ in Ω ;
- 3) $|\text{grad } \omega|_S > 0$

(S is the boundary of the domain Ω), then for it an estimate of the type of A. A. Markov's inequality holds:

$$\max_{\bar{\Omega}} \left| \frac{\partial}{\partial x_j} (\omega P_i) \right| \leq A i^2 \max_{\bar{\Omega}} |\omega P_i|;$$

functions of this form may be used in solving boundary-value problems for differential equations for domains with zero corner points.

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Note: Figure translations are in progress. See original paper for figures.

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