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Abstract

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WEAK ISOMORPHISM OF TRANSFORMATIONS WITH AN INVARIANT MEASURE

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§ 1. The entropy of a transformation with an invariant measure was defined simultaneously with the concept of a K-automorphism in the original paper of A. N. Kolmogorov ([1]). It soon became clear that these concepts are closely connected with one another. There is a hypothesis according to which entropy constitutes a complete system of metric invariants of a K-automorphism. This hypothesis received some confirmation in the work of L. D. Meshalkin ([2]), who gave examples of isomorphism of distinct Bernoulli automorphisms with equal entropy. However, further development of the results of ([2]) encountered difficulties: attempts to extend the method used in ([2]) to any sufficiently broad class of K-automorphisms either do not achieve their aim, or lead to a homomorphism with respect to which it is not possible to determine whether it is an isomorphism.

In the present work the concept of "weak isomorphism" of automorphisms is introduced, possibly broader than the concept of isomorphism. It turns out that the study of K-automorphisms up to weak isomorphism is simpler. Below (§ 4) we indicate a class of K-automorphisms for which entropy constitutes a complete system of invariants of weak isomorphism. The method by which this result was obtained is based on the study of the connection between various perfect partitions (§ 3) associated with a given automorphism. It seems plausible that entropy constitutes a complete system of invariants of weak isomorphism in a much more general case.

As for the basic concepts used below, we refer to the survey ([3]). Let us recall here only some notation. M is a Lebesgue space with measure μ; ε is the partition of the space into individual points; ν is the trivial partition, whose only element is all of M; if ξ is a measurable partition, then $\mathfrak{S}(\xi)$ is the σ-algebra of measurable sets composed mod 0 of elements of the partition ξ; the notation $\xi \leq \eta$, where ξ and η are partitions, means that η is a subpartition of ξ; Π and $\cap$ are the signs of multiplication and intersection of measurable partitions; $H(\xi)$ is the entropy of the measurable partition ξ; $H(\xi \mid \eta)$ is the mean conditional entropy of the measurable partition ξ relative to η; T is an automorphism of the space M; ξ_T^n (shorter ξ^n), ξ_T^- (shorter ξ^-) and ξ_T are the partitions determined by the partition ξ and the automorphism T by the formulas

$$\xi_T^n = \prod_{k=0}^{n} T^k \xi, \qquad \xi_T^- = \prod_{k=0}^{\infty} T^{-k} \xi, \qquad \xi_T = \sum_{k=-\infty}^{\infty} T^k \xi;$$

$h(T,\xi)$ and $h(T)$ are functions defined by the formulas

$$h(T,\xi) = H(T\xi \mid \xi_T^-), \qquad h(T) = \sup_{\xi} h(T,\xi);$$

$h(T)$ is called the entropy of the automorphism T; $\pi(T)$ is the largest partition with zero entropy introduced by M. S. Pinsker ([5]); in the case $\pi(T) = \nu$ the automorphism is called a K-automorphism. Below the case of automorphisms T with $h(T) < \infty$ is studied.

§ 2. The concept of weak isomorphism. The following definition is well known ([3]).

Definition 1. An automorphism T_1 of a space M_1 is called a **homomorphic image** of an automorphism T_2 of a space M_2 if there exists a homomorphism U of the Lebesgue space M_2 onto the Lebesgue space M_1 such that

$$T_1 U = U T_2.$$

If the homomorphism participating in Definition 1 is an isomorphism, then the automorphisms T_1 and T_2 are called **metrically isomorphic**, or simply isomorphic.

The concept of a homomorphic image of an automorphism is closely connected with the concept of a factor-automorphism (see ([3]), p. 13). Recall that the factor-automorphism of an automorphism T with respect to an invariant partition ξ $(T\xi = \xi)$ is called the automorphism T_ξ induced by the automorphism T on the factor-space $M \mid \xi$. Every homomorphic image of a given automorphism is isomorphic to one of its factor-automorphisms.

Definition 2. Automorphisms T_1 and T_2 are called **weakly isomorphic** if T_1 is a homomorphic image of T_2 and T_2 is a homomorphic image of T_1, or, equivalently, if T_1 is isomorphic to some factor-automorphism of T_2 and T_2 is isomorphic to some factor-automorphism of T_1.

Let us show that weakly isomorphic automorphisms are spectrally isomorphic. Let T_1 and T_2 be weakly isomorphic, and let U_1 (U_2) be the unitary operator in $L^2_{M_1}$ $(L^2_{M_2})$ corresponding to T_1 (T_2). If $d\sigma_i(\lambda)$ and $\nu_i(\lambda)$ are, respectively, the maximal spectral type and the multiplicity function of the operator U_i, $i = 1, 2$, then from the fact that T_2 is a homomorphic image of T_1 it follows that $d\sigma_2 \ll d\sigma_1$ and $\nu_2(\lambda) \le \nu_1(\lambda)$ almost everywhere with respect to the measure $d\sigma_1(\lambda)$. Similarly, from the fact that T_1 is a homomorphic image of T_2, we obtain $d\sigma_1 \ll d\sigma_2$ and $\nu_1(\lambda) \le \nu_2(\lambda)$ almost everywhere with respect to the measure $d\sigma_2$. Ultimately we have $d\sigma_1$ equivalent to $d\sigma_2$ and $\nu_1(\lambda) = \nu_2(\lambda)$ almost everywhere with respect to the measure $d\sigma_1$ ([6]).

Further, from the definition of entropy it follows that weakly isomorphic automorphisms have the same entropy. Since the class of automorphisms with completely positive entropy coincides with the class of K-automorphisms ([4]), under weak isomorphism the property of a transformation being a K-automorphism is preserved. In the case of automorphisms with discrete spectrum, weak isomorphism is equivalent to isomorphism.

§ 3. Perfect partitions associated with an automorphism. In ([4]) it was shown that for an arbitrary automorphism T there exists a partition ξ with the following properties:

$$1)\ T\xi \geq \xi; \qquad 2)\ \Pi T^k \xi = \varepsilon; \qquad 3)\ \cap T^k \xi = \pi(T); \qquad 4)\ H(T\xi \mid \xi) = h(T).$$

A partition satisfying these four relations will be called **perfect**. That such partitions must play an essential role in the problem of isomorphism became clear already after the work of A. N. Kolmogorov ([1]). V. A. Rokhlin proposed the following method of obtaining new perfect partitions from a given perfect partition ξ. First, every partition ξ_1 for which the inequalities $T^{-1}\xi \leq \xi_1 \leq \xi$ are fulfilled will also be perfect. Indeed, the fulfillment of relations 1)–3) is obvious. The fulfillment of 4) follows from the chain of equalities

$$H(T\xi_1 \mid \xi_1) = H(T\xi_1 \mid \xi) + H(\xi \mid \xi_1) = H(\xi \mid \xi_1) + H(\xi_1 \mid T^{-1}\xi) =$$

$$= H(\xi \mid T^{-1}\xi) = h(T).$$

Second, one may carry out such a passage to a new perfect partition successively any finite number of times. Third, under certain conditions a limiting passage in such a process is possible.

Below we investigate in more detail the connection between the various perfect partitions corresponding to a given automorphism.

Suppose there is a decreasing sequence of perfect partitions $\xi = \xi_0 \succcurlyeq \xi_1 \succcurlyeq \xi_2 \succcurlyeq \cdots$ such that

$$T^{-1}\xi_i \preccurlyeq \xi_{i+1} \preccurlyeq \xi_i. \tag{1}$$

Form the intersection $\bigcap_{i=0}^{\infty} \xi_i = \xi'$. The question is: under what conditions will this intersection be perfect, and can every perfect partition $\xi' \preccurlyeq \xi$ be obtained as the intersection of a decreasing sequence of perfect partitions satisfying (1)?

Take some finite or countable partition $\eta \preccurlyeq \xi$ and put, for $n > 0$,

$$\xi_n = \eta T^{-n-1}\xi.$$

It is obvious that the inequalities (1) hold for ξ_n, and that all ξ_n are perfect. Let $\eta(\xi) = \bigcap_n \xi_n$. As the properties of the partition $\eta(\xi)$ shown below indicate, it is very similar to the partition η^-. From the construction it is immediately clear that $\eta(\xi) \succcurlyeq \eta^-$, $T\eta(\xi) = T\eta \cdot \eta(\xi)$. Less trivial is the equality

$$H(T\eta \mid \eta(\xi)) = h(T, \eta). \tag{2}$$

It can be obtained analogously to Lemma 1 of paper ([4]). From (2) and from the properties of entropy it is not hard to derive that then, on almost every element $C(\eta^-)$ of the partition η^-, the partitions η^k and $\xi(\eta)$ are independent for every $k > 0$. As a consequence of the last assertion, it turns out that if the inequality $\xi(\eta) > \eta^-$ holds, then the inequality $\prod T^k \xi(\eta) > \eta_\tau$ also holds.

Theorem 1. For every perfect partition $\xi' \preccurlyeq \xi$ there exists a partition $\eta \preccurlyeq \xi$, at most countable, such that $\xi' = \eta(\xi)$.

Proof. From the condition $h(T) < \infty$ one can derive that there exists an at most countable partition $\eta \preccurlyeq \xi' \preccurlyeq \xi$, $\eta \in Z$, for which

$$T\xi' = T\eta \cdot \xi'.$$

It is easy to see that, for every $n > 0$,

$$\xi' \preccurlyeq \eta T^{-n-1}\xi.$$

Therefore $\xi' \preccurlyeq \eta(\xi)$. If in the last relation there were a strict inequality, then from the properties of the partitions $\eta(\xi)$ it would follow that $\prod_k T^k \xi' < \prod_k T^k \eta(\xi)$, which is impossible, since $\prod_k T^k \xi' = \varepsilon$ by virtue of the perfection of ξ'. The theorem is proved.

It follows from Theorem 1 that every perfect partition $\xi' \preccurlyeq \xi$ can be obtained as the intersection of perfect partitions satisfying (1). Indeed, as was already indicated, the inequalities (1) are obvious for the partitions $\xi_n = \eta T^{-n-1}\xi$, $n > 0$, and $\xi' = \bigcap_n \xi_n$.

Let us now consider the question of what the partition η must be in order that $\eta(\xi)$ be perfect. Equality (2) shows that for this it is necessary that $h(T, \eta) = h(T)$. It turns out that under some conditions it is also sufficient. The following propositions are true:

I. If $H(T\eta(\xi) \mid \eta(\xi)) = h(T)$ and $\sum_n H(\xi_n \mid \xi_{n+1}) < \infty$, then $\eta(\xi)$ is perfect.

II. In order that $H(T\eta(\xi) \mid \eta(\xi)) = h(T)$, it is necessary and sufficient that

$$\lim_{n\to\infty} H(\xi_n \mid \xi_{n+1}) = 0.$$

It is known ([1]) that there exist partitions satisfying relations 1)–3) of an extremal partition and not satisfying relation 4). However, among partitions not exceeding some perfect partition this cannot occur.

III. If ξ is a perfect partition and the partition $\xi' \preccurlyeq \xi$ satisfies relations 1)–3) of a perfect partition, then ξ' is perfect.

Concerning Proposition III, let us note that it is interesting to clarify how the existence of a countable generating partition for the completion ξ' affects the existence of a countable generating partition for ξ'. It would also be interesting to extend the results given here to the case of arbitrary intersections of complete partitions.

The results of the present section are used essentially in the proof of the main theorem of § 4.

§ 4. **The main theorem.** Let the space M consist of sequences $x = \{x_n\}$, infinite in both directions, where each x_n assumes a countable number of values $1, 2, \ldots$. Define a measure μ in M, by means of numbers q_i, $q_i \geqslant 0$, $\sum_i q_i = 1$, by the formula

$$\mu\{x : \ x_{n_0} = i_0, x_{n_0+1} = i_1, \ldots, x_{n_0+k} = i_k\} = q_{i_0} q_{i_1} \cdots q_{i_k}.$$

In other words, the space M may be regarded as the space of realizations of a certain sequence of independent random variables. The shift transformation T : $Tx = x' = \{x'_n\}$, $x'_n = x_{n-1}$, preserves the measure μ and is an automorphism of the space M. Such an automorphism is called a Bernoulli automorphism ([3]). Bernoulli automorphisms are classical examples of K-automorphisms.

Theorem 2. *Let T be an arbitrary ergodic automorphism of some Lebesgue space M. Every Bernoulli automorphism T_1 for which $h(T) \geqslant h(T_1)$ is isomorphic to some factor-automorphism of the automorphism T. More precisely, if ξ is a partition of the space M such that $T\xi \geqslant \xi$, $H(T\xi \mid \xi) \geqslant h(T_1)$, then one can find a partition $\eta \leqslant \xi$ of the space M into sets $A_1, A_2, \ldots$, for which*

$$\mu\{A_{i_0} \cap TA_{i_1} \cap \cdots \cap T^k A_{i_k}\} = q_{i_0} q_{i_1} \cdots q_{i_k} \quad \text{for all } k;\ i_0, i_1, \ldots, i_k.$$

We give consequences of Theorem 2.

1. *Bernoulli automorphisms with equal entropy are weakly isomorphic.*
2. *Every arbitrary ergodic automorphism with positive entropy has an infinite-dimensional invariant subspace in which there is an everywhere dense set of functions satisfying the central limit theorem.*
3. *For an ergodic endomorphism T, every Bernoulli endomorphism* T_1 for which $h(T) = h(T_1)$ is its factor-endomorphism.*

Consequence 3 was pointed out to me by V. A. Rokhlin. Consequences 4 and 5 answer certain questions posed in ([3]).

4. *Every arbitrary ergodic automorphism with positive entropy has a factor-automorphism with the same entropy which is a K-automorphism.*

5. *For every arbitrary ergodic automorphism with finite entropy there exists a partition η such that $H(\eta) = h(T, \eta) = h(T)$.*

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[*] By a Bernoulli endomorphism we mean such an endomorphism whose natural extension ([7]) is a Bernoulli automorphism.

Note: Figure translations are in progress. See original paper for figures.