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Abstract

Full Text

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE NOMOGRAPHABILITY OF FUNCTIONS OF MANY VARIABLES

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Let $f_{ik} = f_{ik}(x_i)$, $i, k = 1, \dots, n$, be the elements of the first n rows of a Massau matrix [1] of order $(n+1)$, whose last column is filled only with ones. Let the elements of the $(n+1)$-st row of this matrix Δ be $g_1(z), \dots, g_n(z), 1$, and let $\omega_1, \dots, \omega_n, \omega_{n+1}$ be their cofactors, i.e., the corresponding minors of order n of the matrix $\overline{\Delta}$, taken with the appropriate signs.

Denote

$$U_i = \sum_{k=1}^{n} f'_{ik}\omega_k, \qquad i = 1, \dots, n, \tag{1}$$

assuming that all f_{ik} have second continuous derivatives in the n-dimensional closed parallelepiped G with edges parallel to the coordinate axes $x_1, \dots, x_n$, and that none of the functions U_i vanishes anywhere in G.

Definition 1. The **Gronwall functions** of the Massau matrix $\overline{\Delta}$ are

$$C_{ik} = \begin{cases} \dfrac{\partial}{\partial x_i} \ln \dfrac{U_i}{U_k^2}, & \text{for } i \neq k, \\ 0, & \text{for } i = k, \end{cases} \tag{2}$$

defined and continuous everywhere in G.

We also introduce the associated functions

$$z_{ik} = \frac{1}{3}\left(2\frac{\partial C_{ki}}{\partial x_i} + \frac{\partial C_{ik}}{\partial x_k}\right), \tag{3}$$

which, as is clear from the obvious equality

$$z_{ik} = -\frac{\partial^2}{\partial x_i \partial x_k} \ln U_i, \qquad i \neq k, \tag{4}$$

are defined and continuous in G.

For brevity, a Massau matrix will be called **admissible** if it satisfies the restrictions formulated above.

Lemma 1. *The Gronwall functions of any Massau matrix admissible in the parallelepiped G satisfy the differential equations*

$$\frac{\partial C_{si}}{\partial x_k} + (1+\delta_{si})z_{sk} = \frac{\partial C_{ki}}{\partial x_s} + (1+\delta_{ki})z_{ks}, \tag{5}$$

$$\frac{\partial z_{ik}}{\partial x_k} = C_{ki}z_{ik}, \qquad i,k,s = 1,\dots,n. \tag{6}$$

Proof. The validity of (5) follows immediately from the definitions. To prove (6), it suffices to note that, for $i \neq k$, by virtue of (4),

$$V_{ik} = z_{ik}\frac{U_i^2}{U_k}, \qquad i,k = 1,\dots,n,$$

is the determinant of the matrix obtained from $\bar{\Delta}$ by replacing the elements of the $(n+1)$-st row by the first derivatives and the elements of the k-th row by the second derivatives of the corresponding elements of the i-th row of this matrix $\bar{\Delta}$. Consequently, V_{ik} does not depend on x_k, whence (6) follows.

Lemma 2. For functions Γ generated by any admissible Massau matrix in G, for $i \neq s$ and $k \neq s$ the relations

$$2\frac{\partial}{\partial x_s}(C_{si} - C_{sk}) = C_{si}^2 - C_{sk}^2 \tag{7}$$

hold.

Proof. Denote by V_{iks} the determinant of the matrix obtained from the matrix $\bar{\Delta}$ by replacing in it the elements of the $(n+1)$-st row by the first derivatives of the corresponding elements of the k-th row ($k \leq n$), and the elements of the s-th ($s \leq n$) row by the first derivatives of the corresponding elements of the i-th ($i \leq n$) row of the same matrix $\bar{\Delta}$.

It is obvious that, for $s \neq i$ and $s \neq k$, V_{iks} does not depend on x_s.

Replace in $\bar{\Delta}$ the elements of the $(n+1)$-st row by the first derivatives of the corresponding elements of the k-th row, and border the resulting matrix by an $(n+2)$-nd row $f'_{i1},\dots,f'_{in},0,0$ and an $(n+2)$-nd column whose elements, except for the last two, are identically zero and are respectively equal to the elements of the r-th ($r \leq n+1$) column, taken with the opposite sign. The determinant of such a matrix for $r \leq n$ is equal to $f'_{ir}U_k - f'_{kr}U_i$, and for $r = n+1$ is identically equal to zero. Expansion with respect to the elements of the last column gives

$$f'_{ir}U_k - f'_{kr}U_i = \sum_{s=1}^{n} f_{sr}V_{iks}, \tag{8}$$

$$\sum_{s=1}^{n} V_{iks} = 0. \tag{9}$$

It is now easily proved that

$$V_{iks} = \frac{U_{iU}k}{U_s}\left(\frac{\partial}{\partial x_s}\ln\frac{U_k}{U_i} + \delta_{is}\frac{\partial}{\partial x_s}\ln U_i - \delta_{ks}\frac{\partial}{\partial x_s}\ln U_k\right). \tag{10}$$

Indeed, if $s = i$ or $s = k$, then the validity of formula (10) is obvious. For $s \neq i$ and $s \neq k$, denoting $U_{ks} = \partial U_k/\partial x_s$, consider the determinant $U_{ks}U_i - U_{is}U_k$ of the matrix obtained from $\bar{\Delta}$ by replacing the elements of the $(n{+}1)$-st row by the first derivatives of the corresponding elements of the s-th row ($s \leq n$), and by replacing the first n elements of the s-th row respectively by $f'_{ir}U_k - f'_{kr}U_i$, $r = 1, \ldots, n$; the $(n+1)$-st element of the s-th row must be replaced by zero. This determinant is transformed with the aid of (8) and (9), which entails (10). From (10), by partial differentiation with respect to x_s, one obtains (7).

Definition 2. A **nomographic representation** in G is any equation

$$\Delta(x_1, \ldots, x_n, z) = \mathrm{Det}\,\bar{\Delta}(x_1, \ldots, x_n, z) = 0, \tag{11}$$

where $\bar{\Delta}$ is some admissible Massau matrix.

Let the left-hand side of the equation

$$z = \varphi(x_1, \ldots, x_n) \tag{12}$$

possesses in G all continuous partial derivatives of second order, and its first derivatives do not vanish in G.

Definition 3. Equation (12) is called **nomographable** in G if there exists a nomographic representation such that

$$\Delta(x_1, \ldots, x_n; \varphi(x_1, \ldots, x_n)) \equiv 0. \tag{13}$$

Just as in (1), one can show that any nomographic representation of the function (12) defines this function as an implicit function of $(x_1, \ldots, x_n) \in G$.

Theorem 1. *If the function (12) is nomographable in the parallelepiped G, then for any nomographic representation of it the functions of Goursat of the corresponding Massau matrix are connected by the identities*

$$\varphi_i\varphi_k(\varphi_i C_{ki} + \varphi_k C_{ik}) = \varphi_i^2\varphi_{kk} - 2\varphi_i\varphi_k\varphi_{ik} + \varphi_k^2\varphi_{ii}, \qquad 1 \le i < k \le n. \quad (14)$$

Proof. Introduce the notation

$$U_k^* = U_1 \dots U_{k-1}U_{k+1} \dots U_n, \quad (15)$$

$$\theta = \left(\sum_{k=1}^{n} \varphi_k U_k^*\right)^{-1}, \quad (16)$$

where it can be shown that everywhere in G the function $\sum_{k=1}^{n} \varphi_k U_k^*$ does not vanish.

For any Massau matrix of every nomographic representation of equation (12), everywhere in G along the graph of the function (12) the identities

$$g_i(z) = \theta \sum_{k=1}^{n} f_{ki}\varphi_k U_k^* \quad (17)$$

hold.

Using the identities

$$\partial(g_r, \varphi)/\partial(x_i, x_k) \equiv 0, \quad (18)$$

transforming the latter by means of (8), (9), and taking into account that the Goursat functions are invariant with respect to the projective group of transformations of the nomogram space, we obtain the required result. It should also be noted here that the system of identities (14) and the system of identities (18) are equivalent.

Taken together, equalities (5), (6), (7), and (14) constitute a necessary condition for nomographability.

For $n = 2$ there are no equalities of type (7); the two equalities of type (5) are equivalent, by the equations (3) defining the adjoint functions; nontrivial equalities of this type are absent. Finally, the two equalities (6) and one equality (14) together give the well-known Goursat criterion.

To prove sufficiency, we shall consider (2) and (4), for given C_{ik} and z_{ik}, as second-order differential equations with respect to f_{ik}, adding certain initial conditions.

Definition 4. Initial conditions at the point $(x_1^{(0)}, \dots, x_n^{(0)})$ are called **normal** if the values of all f_{ik} and of their first derivatives are specified so that all U_i

are different from zero at this point and there exist n real numbers λ_i such that, for any $s \neq i$, $1 \leq s \leq n$, at the point $(x_1^{(0)}, \dots, x_n^{(0)})$ one has

$$2\frac{\partial}{\partial x_i} \ln U_s + C_{is} = \lambda_i, \qquad i = 1, \dots, n. \tag{19}$$

Lemma 3. If, in a neighborhood of the point $(x_1^{(0)}, \dots, x_n^{(0)})$, continuous functions C_{ik}, z_{ik} are given, with all $C_{ii} = 0$, satisfying the differential equations (5), (6), (7), then, for every choice of normal initial conditions, there exists a unique system of functions f_{ik}, $i, k = 1, \dots, n$, twice continuously differentiable in some neighborhood of $(x_1^{(0)}, \dots, x_n^{(0)})$ and satisfying the differential equations (2) and (4).

Proof proceeds along the same lines as the proof of the corresponding Lemma 4 in ([1]). We note that the relations (19), imposed on the initial values, make it possible to express all these initial values in terms of $n^2 + 2n$ independent arbitrary parameters, i.e., in other words, the nomogram is determined up to a projective transformation of its space.

If one agrees not to distinguish projectively equivalent nomograms from one another and assumes that C_{ik} and z_{ik} are given in some parallelepiped G, inside which lies the point $(x_1^{(0)}, \dots, x_n^{(0)})$, then the solution constructed in some neighborhood of this point can be continued to the boundary of G.

Theorem 2. If in the parallelepiped G there exists a continuous solution of the system of differential equations with respect to the Ronval functions, consisting of equations (5), (6), (7) with the relations (14) between these functions, then the function (12) is nomographable in the parallelepiped G.

Proof. By Lemma 3 one can construct the first n rows of the Massau matrix. The elements of the last row are found from formulas (17).

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REFERENCES

1. S. V. Smirnov, DAN, **124**, 34 (1959).

Note: Figure translations are in progress. See original paper for figures.