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Abstract

Full Text

Mathematics

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DISTRIBUTION OF PRIMITIVE INTEGER POINTS ON CERTAIN CONES

(Presented by Academician I. M. Vinogradov on 22 XI 1961)

Let there be a surface of second order

$$f(x, y, z) = 0.$$

We assume that the coefficients of the quadratic form $f(x, y, z)$ are rational integers. A solution (x, y, z) of this equation in rational integers satisfying $(x, y, z) = 1$ will be called a primitive integer point. ( ) denotes the greatest common divisor.

Theorem 1. *For the number of primitive integer points $A(h)$ lying on the surface of the cone $x^2 + y^2 - z^2 = 0$ in the region $0 < z \leq h$, the following asymptotic formula holds: as $h \to \infty$,*

$$A(h) = \frac{4}{\pi} h + O(\sqrt{h}).$$

Proof. The problem of the number of primitive integer points $A(h)$ lying on the surface of the cone $x^2 + y^2 - z^2 = 0$ in the region $0 < z \leq h$ we reduce to the problem of primitive integer solutions of the equation

$$x^2 + y^2 = z^2.$$

The following lemmas are known (see, for example, (1), problem 9a, pp. 23, 116, and question 176, chap. II, pp. 36, 127):

Lemma 1. *All primitive integer solutions of the equation $x^2 + y^2 = z^2$ with $x \geq 0,\ y \geq 0,\ z > 0$ are obtained uniquely from the formulas:*

$$\begin{cases} x = 2mn, \\ y = m^2 - n^2, \\ z = m^2 + n^2, \end{cases} \qquad m > 0,\ n \geq 0,\ m > n,\ (m, n) = 1,$$

$$m \text{ and } n \text{ of different parity}$$

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Fig. 1

Figure 1: Fig. 1

$$\begin{cases} x = m^2 - n^2, \\ y = 2mn, \\ z = m^2 + n^2, \end{cases} \qquad m > 0,\ n \geq 0,\ m > n,\ (m, n) = 1,$$

m and n of different parity.

Lemma 2. *Let $k > 1$ and suppose systems are given*

$$x_1', x_2', \dots, x_k'; \quad x_1'', x_2'', \dots, x_k''; \ \dots; \quad x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)},$$

each of which consists of integers not all simultaneously zero. Suppose, further, that for these systems a function $f(x_1, x_2, \dots, x_k)$ is uniquely defined. Then

$$S' = \sum \mu(d) S_d,$$

where S' denotes the sum of the values $f(x_1, x_2, \dots, x_k)$ extended over systems of relatively prime numbers, and S_d denotes the sum of the values $f(x_1, x_2, \dots, x_k)$ extended over systems of numbers simultaneously divisible by d. Here d runs through the positive integers.

From Lemma 1 it follows that

$$A(h) = 4M(h) - 4,$$

where $M(h)$ is the number of integer points lying inside the quarter circle $m^2 + n^2 \leq h,\ m \geq 0,\ n \geq 0$, with $(m, n) = 1$, and with m and n of different parity.

Finally, using Lemma 2, we reduce the problem of the number $M(h)$ of integer points with the indicated conditions to the problem of the number of integer points inside a circle. Applying the known estimate for the number $K(R)$ of integer points inside the circle $x^2 + y^2 \leq R$,

$$K(R) = \pi R + O(R^{1/3}),$$

we obtain the required result.

Fig. 1

By means of the method used in the proof of Theorem 1, one can prove a theorem on the distribution of primitive integer points inside a sector of aperture φ lying on the surface of the cone $x^2 + y^2 - z^2 = 0$ (Fig. 1).

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Theorem 2. *Let $A(h, \varphi)$ be the number of primitive integer points lying on the surface of the cone $x^2 + y^2 - z^2 = 0$ in the region under consideration. Then, as $h \to \infty$,*

$$A(h, \varphi) = \frac{2\theta}{\pi^2} h + O(\sqrt{h} \ln h),$$

where φ and θ are connected by the relation

$$2 \cos \varphi - \cos \theta = 1.$$

Theorem 1 can be generalized. Denote by $F(h)$ the number of primitive integer points lying on the surface of the cone

$$u^2 + Av^2 = w^2$$

in the region $0 < w \leq h$, where A is a positive integer not divisible by the square of any integer other than 1.

Theorem 3. *As $h \to \infty$,*

$$F(h) = \begin{cases} \dfrac{2^{n+2}\sqrt{A}}{\pi \prod\limits_{i=1}^{n} (A_i + 1)} h + O(\sqrt{h} \ln h), & \text{if } A = \prod\limits_{i=1}^{n} A_i,\ A_i \neq 1, 2; \\ \dfrac{2^{m+1}\sqrt{A}}{\pi \prod\limits_{i=1}^{m} (A'_i + 1)} h + O(\sqrt{h} \ln h), & \text{if } A = 2\prod\limits_{i=1}^{m} A'_i,\ A'_i \neq 1, 2; \\ \dfrac{2\sqrt{2}}{\pi} h + O(\sqrt{h} \ln h), & \text{if } A = 2. \end{cases}$$

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REFERENCES CITED

1. I. M. Vinogradov, *Fundamentals of Number Theory*, Moscow, 1952.

Note: Figure translations are in progress. See original paper for figures.

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