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Abstract

Full Text

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APPROXIMATION OF PERIODIC FUNCTIONS BELONGING TO THE CLASSES $W_{p,\theta}^{(r_1,r_2)}$ BY FOURIER SUMS

(Presented by Academician A. I. Mal' tsev on 8 VI 1962)

Let $f(x, y)$ be a periodic function and

$$f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{kl} e^{i(kx+ly)}$$

its Fourier series.

We define the partial derivative of arbitrary (including fractional) order $r > 0$, in the Weyl sense, of f with respect to the variable x as follows:

$$f_x^{(r)}(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \gamma_k^{(r)} a_{kl} e^{i(kx+ly)},$$

where $\gamma_k^{(r)}$ is a single-valued function determined by the equalities

$$\gamma_k^{(r)} = e^{r\pi i/2} k^r, \quad \gamma_{-k}^{(r)} = e^{-r\pi i/2} k^r \quad (k = 0, 1, 2, \dots).$$

In the same way the mixed derivative is defined:

$$f_{yx}^{(r_2,r_1)} = f_{xy}^{(r_1,r_2)} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \gamma_k^{(r_1)} \gamma_l^{(r_2)} a_{kl} e^{i(kx+ly)},$$

which does not depend on the order of differentiation.

Definition. Let r_1 and r_2 be given positive, not necessarily integer, numbers. By definition, a function f , 2π -periodic in each of the variables, belongs to $W_{p,\theta}^{(r_1,r_2)}$ if $f \in L_p$ and has the property that the function

$$\frac{\partial^{r_1} f}{\partial x^{r_1}} + e^{i\theta} \frac{\partial^{r_2} f}{\partial y^{r_2}} = \varphi(x, y) \quad (1)$$

is summable in the p -th power ($1 \leq p < \infty$), and for its norm the inequality holds

$$\|\varphi\|_p = \left(\int_0^{2\pi} \int_0^{2\pi} |\varphi(x, y)|^p dx dy \right)^{1/p} \leq 1. \quad (2)$$

In this connection one may consider only those pairs r_1 and r_2 for which the differential equation (1) has a unique periodic solution. It can be shown that the uniqueness property holds if and only if, for every k ,

$$\begin{aligned} \theta + \frac{(r_1 + r_2)\pi}{2} &\neq (2k + 1)\pi, & \theta - \frac{(r_1 + r_2)\pi}{2} &\neq (2k + 1)\pi, \\ \theta + \frac{(r_1 - r_2)\pi}{2} &\neq (2k + 1)\pi, & \theta - \frac{(r_1 - r_2)\pi}{2} &\neq (2k + 1)\pi. \end{aligned} \quad (3)$$

$$(k = 0, \pm 1, \dots)$$

Here we shall consider only the values $\theta = 0, \pi$. For them the uniqueness property holds. Let

$$s_{m,n} = s_{m,n}(f, x, y) = \sum_{-(m-1)}^{(m-1)} \sum_{-(n-1)}^{(n-1)} \frac{b_{kl}}{(ik)^{r_1} + e^{i\theta}(il)^{r_2}} e^{i(kx+ly)}$$

be the m, n -th Fourier sum of the function f . Denote by

$$\varepsilon_{m,n}(W_{\infty, \theta}^{(r_1, r_2)}) = \sup_{f \in W_{\infty, \theta}^{(r_1, r_2)}} \|f - s_{m,n}\|_{\infty};$$

$$\varepsilon_{m,n}(W_{1, \theta}^{(r_1, r_2)}) = \sup_{f \in W_{1, \theta}^{(r_1, r_2)}} \|f - s_{m,n}\|_1$$

the upper bounds for the deviations of functions f from their Fourier sums $s_{m,n}$, extended to the classes $W_{p, \theta}^{(r_1, r_2)}$ for $p = \infty$ and $p = 1$. Introduce the number

$$\lambda = \frac{|\vartheta_{-m,n}^{(r_1, r_2)}|}{|\vartheta_{m,n}^{(r_1, r_2)}|},$$

depending on r_1, r_2, m, n , where

$$\vartheta_{m,n}^{(r_1,r_2)} = \frac{1}{(im)^{r_1} + e^{i\theta}(in)^{r_2}},$$

and put $\lambda_1 = \frac{1}{\lambda}$.

Theorem 1. The following asymptotic formula is valid:

$$\begin{aligned} \varepsilon_{m,n} = \varepsilon_{m,n}(W_{\infty,\theta}^{(r_1,r_2)}) &= \varepsilon_{m,n}(W_{1,\theta}^{(r_1,r_2)}) = \frac{16E + 8(\lambda^2 - 1)F}{\pi^4 |m^{r_1} + e^{i\alpha}n^{r_2}|} \ln m \ln n + \\ + O \left\{ \frac{\ln m [16E + 8(\lambda^2 - 1)F]}{m^{r_1}} + \frac{\ln n [16E + 8(\lambda^2 - 1)F]}{n^{r_2}} \right\} &+ O \left(\frac{\ln m}{m^{r_1}} + \frac{\ln n}{n^{r_2}} \right), \end{aligned}$$

where

$$\alpha = \theta + \frac{r_2 - r_1}{2}\pi,$$

$$E = E \left(\lambda, \frac{\pi}{2} \right) = \int_0^{\pi/2} \sqrt{1 - \lambda^2 \cos^2 t} dt \quad \text{for } 0 < \lambda < 1,$$

$$F = F \left(\lambda, \frac{\pi}{2} \right) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - \lambda^2 \cos^2 t}};$$

$$\begin{aligned} \varepsilon_{m,n} = \frac{16E + 8(\lambda_1^2 - 1)F}{\pi^4 |m^{r_1} + e^{i\beta}n^{r_2}|} \ln m \ln n + O \left\{ \frac{\ln m [16E + 8(\lambda_1^2 - 1)F]}{m^{r_1}} + \right. \\ \left. + \frac{\ln n [16E + 8(\lambda_1^2 - 1)F]}{n^{r_2}} \right\} + O \left(\frac{\ln m}{m^{r_1}} + \frac{\ln n}{n^{r_2}} \right), \end{aligned}$$

where

$$\beta = \theta + \frac{r_1 + r_2}{2}\pi \quad \text{for } 0 < \lambda_1 < 1;$$

$$\varepsilon_{m,n} = \frac{16 \ln m \ln n}{\pi^4 |m^{r_1} + e^{i\alpha}n^{r_2}|} + O \left(\frac{\ln m}{m^{r_1}} + \frac{\ln n}{n^{r_2}} \right) \quad \text{for } \lambda = \lambda_1 = 1.$$

In the one-dimensional case, for integer r , this asymptotic estimate was obtained by A. N. Kolmogorov ⁽¹⁾; in the fractional case (for all $r > 0$) by V. T. Pinkevich

(²). In the two-dimensional case with $r_1 = r_2 = 2$, this estimate was obtained by Ya. S. Bugrov (³).

Below we give the proof, which proceeds for $\lambda \neq 1$. In the case $\lambda = 1$ it is modified somewhat. We note that the case $\lambda = 1$ occurs when r_1 and r_2 are even and satisfy conditions (3).

Idea of the proof. From the periodicity of f it follows that the equality

$$\int_0^{2\pi} \int_0^{2\pi} \varphi(x, y) dx dy = 0 \quad (4)$$

holds.

We obtain an integral representation for $f(x, y)$. To this end, expand the function $\varphi(x, y)$ in a Fourier series

$$\varphi(x, y) \sim \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} ' b_{kl} e^{i(kx+ly)}.$$

The prime on the sum means that the term $k = l = 0$ is absent. Then

$$f(x, y) = \frac{b_{0,0}}{4} + \frac{1}{4\pi^2} \int_D K^{(r_1, r_2)}(x-u, y-v) a(u, v) du dv, \quad (5)$$

where $D = \{0 \leq x, y \leq 2\pi\}$ and

$$K^{(r_1, r_2)}(u, v) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} ' \frac{e^{i(ku+lv)}}{(ik)^{r_1} + e^{i\theta}(il)^{r_2}}.$$

Putting

$$\Psi_{m,n}^{(r_1, r_2)}(u, v) = K^{(r_1, r_2)}(u, v) - \sum_{-(m-1)}^{(m-1)} ' \sum_{-(n-1)}^{(n-1)} ' \frac{e^{i(ku+lv)}}{(ik)^{r_1} + e^{i\theta}(il)^{r_2}},$$

we obtain

$$R_{m,n}(f) = f(x, y) - s_{m,n}(f) = \frac{1}{4\pi^2} \int_D \Psi_{m,n}^{(r_1, r_2)}(x, -u, y-v) a(u, v) du dv.$$

Carrying out Abel's transformation successively four times, we obtain the following expression for $\Psi_{m,n}^{(r_1, r_2)}(u, v)$:

$$\Psi_{m,n}^{(r_1,r_2)}(u,v) = A_{m,n}^{(r_1,r_2)}(u,v) + Q_{m,n}(u,v),$$

where

$$A_{m,n}^{(r_1,r_2)}(u,v) = \frac{1}{2 \sin \frac{u}{2} \sin \frac{v}{2}} \left\{ \left| \vartheta_{m,n}^{(r_1,r_2)} \right| \cos \left(\frac{2m-1}{2}u + \frac{2n-1}{2}v + \psi \right) + \left| \vartheta_{-m,n}^{(r_1,r_2)} \right| \cos \left(\frac{2m-1}{2}u - \frac{2n-1}{2}v + \psi_1 \right) \right\}; \quad (6)$$

$Q_{m,n}(u,v)$ is a certain remainder containing all nonprincipal terms of the expansion, and

$$\operatorname{tg} \psi = \frac{m^{r_1} \cos \frac{r_1 \pi}{2} + n^{r_2} \cos \frac{r_2 \pi}{2}}{m^{r_1} \sin \frac{r_1 \pi}{2} + n^{r_2} \sin \frac{r_2 \pi}{2}}, \quad \operatorname{tg} \psi_1 = -\frac{m^{r_1} \cos \frac{r_1 \pi}{2} + n^{r_2} \cos \frac{r_2 \pi}{2}}{m^{r_1} \sin \frac{r_1 \pi}{2} + n^{r_2} \sin \frac{r_2 \pi}{2}}.$$

The equality

$$\frac{1}{4\pi^2} \int_{\Delta} \left| A_{m,n}^{(r_1,r_2)}(u,v) \right| du dv = \frac{\ln m \ln n}{\pi^4} \left| \vartheta_{m,n}^{(r_1,r_2)} \right| [16E + 8(\lambda^2 - 1)F] + o \left\{ \frac{\ln m [16E + 8(\lambda^2 - 1)F]}{m^{r_1}} + \frac{\ln n [16E + 8(\lambda^2 - 1)F]}{n^{r_2}} \right\}. \quad (7)$$

Indeed, let us partition the domain

$$\Delta_i^j = \left\{ \frac{4\pi}{2m-1} \leq u \leq 2\pi - \frac{4\pi}{2m-1}; \frac{4\pi}{2n-1} \leq v \leq 2\pi - \frac{4\pi}{2n-1} \right\}$$

into elementary parallelograms by two families of straight lines

$$\frac{2m-1}{2}u + \frac{2n-1}{2}v = k_1; \quad \frac{2m-1}{2}u - \frac{2n-1}{2}v = k_2 \quad (k_1, k_2 = \text{const}),$$

inside which each of the terms standing in the braces of formula (6) does not change sign. Estimating the resulting integrals with the aid of the generalized mean-value theorem and finding the order of the integrals over the incomplete parallelograms along the entire boundary strip discarded when approximating Δ by parallelograms, we obtain (7).

Using equality (7), one can prove the required asymptotic formula in the norm L_∞ .

Applying the reasoning of Ya. S. Bugrov⁽³⁾, one can prove the asymptotic formula also in the norm L_1 .

The formula proved is asymptotic if, for example, $c_1 \leq m^{r_1}/n^{r_2} \leq c_2$, where c_1 and c_2 are positive constants, since in this case the first term on the right-hand side of this formula is the principal one.

Similar asymptotic estimates can be carried out for classes of equations, examples of which are the equations

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \pm \frac{\partial^4 f}{\partial x^2 \partial y^2} = \varphi(x, y).$$

In conclusion we give a theorem.

Theorem 2. Let $f \in L_p$ ($1 < p < \infty$), and let f be a 2π -periodic function in each of the variables.

If

$$\frac{\partial^{k_1+\dots+k_m} f}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} \in L_p, \quad \frac{\partial^{k_{m+1}+\dots+k_n} f}{\partial x_{m+1}^{k_{m+1}} \dots \partial x_n^{k_n}} \in L_p$$

and, moreover,

$$\frac{\partial^{\theta_1 k_1 + \dots + \theta_m k_m} f}{\partial x_1^{\theta_1 k_1} \dots \partial x_m^{\theta_m k_m}} \in L_p, \quad \frac{\partial^{\theta_{m+1} k_{m+1} + \dots + \theta_n k_n} f}{\partial x_{m+1}^{\theta_{m+1} k_{m+1}} \dots \partial x_n^{\theta_n k_n}} \in L_p,$$

where θ_i is equal to 1 or 0, $i = 1, 2, \dots, n$.

Then

$$\frac{\partial^{\lambda(k_1+\dots+k_m)+\mu(k_{m+1}+\dots+k_n)} f}{\partial x_1^{\lambda k_1} \dots \partial x_m^{\lambda k_m} \partial x_{m+1}^{\mu k_{m+1}} \dots \partial x_n^{\mu k_n}} \in L_p.$$

where $\lambda, \mu \geq 0$, $\lambda + \mu \leq 1$.

The theorem can be proved by means of the Marcinkiewicz multiplier method⁽⁴⁾.

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Note: Figure translations are in progress. See original paper for figures.

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