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Abstract

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ASYMPTOTIC REPRESENTATION OF FUNDAMENTAL SOLUTIONS OF HYPOELLIPTIC EQUATIONS AND A PROBLEM IN THE WHOLE SPACE WITH CONDITIONS AT INFINITY

(Presented by Academician I. G. Petrovskii on February 15, 1962)

§ 1. **Introduction.** In this paper we consider equations

$$P\left(i\frac{\partial}{\partial x}\right)u(x)=f(x), \tag{1}$$

where $x=(x_1,x_2,\dots,x_n)$, $P\left(i\frac{\partial}{\partial x}\right)=P\left(i\frac{\partial}{\partial x_1},i\frac{\partial}{\partial x_2},\dots,i\frac{\partial}{\partial x_n}\right)$ is a hypoelliptic operator ([1]) in n variables with constant real coefficients, acting on functions defined in the whole n-dimensional real space R_x^n.

By W we shall denote a class of functions, defined in the whole space R_x^n, in which there exists a unique solution of equation (1). Usually the classes W are readily found after the asymptotics at infinity of the fundamental solutions for equation (1) have been obtained.

For $n=2$, the asymptotics at infinity of the fundamental solutions and the classes W were obtained by the author for a fairly broad class of hypoelliptic operators ([2]). Sommerfeld ([3]), I. N. Vekua ([4]), and B. P. Paneiakh ([5]) found the asymptotics of fundamental solutions and the classes W for certain particular types of equations (1) with $n>2$. In the work of V. P. Palamodov ([6]) the class W is found for any equation (1), not necessarily hypoelliptic, but whose characteristic polynomial $P(s)$, $s=(s_1,s_2,\dots,s_n)$, has no real zeros.

In the present paper we shall find the asymptotics at infinity of certain fundamental solutions and the classes W for any hypoelliptic operator satisfying the following four conditions:

1. $P(s)$ has only real coefficients.

2. $P(s)$ has real zeros.

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3. grad $P(s) \neq 0$ at the real zeros of $P(s)$.

From the listed conditions it follows that, in the n-dimensional real space R_s^n, the real zeros of the polynomial form several smooth closed surfaces. We denote them by K_j, $j = 1, 2, \dots, m$.

4. The total curvature $K(s)$ of the surfaces K_j, $j = 1, 2, \dots, m$, is everywhere nonzero.*

The method of the present paper is a generalization of the method used by the author in ([2]).

§ 2. **Construction of a fundamental solution.** A fundamental solution for an operator satisfying the listed conditions will be obtained in the form of the integral

$$E(x) = \frac{1}{(2\pi)^n} \int_H \frac{\exp[-i(x, s)]}{P(s)}\, dH, \qquad (2)$$

where H is a set analogous to Hörmander' s ladder ([7]). We shall now construct the set H.

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* By the total curvature K of a surface we mean the ratio of the determinant of the second quadratic form of the surface to the determinant of the first quadratic form.

Let α be some orthogonal matrix with real coefficients, let $s(\alpha) = (s_1(\alpha), s_2(\alpha), \dots, s_n(\alpha))$ be the coordinate system obtained from $s = (s_1, s_2, \dots, s_n)$ after the transformation with matrix α, and let $R(\alpha)$ be the subspace of the subspace R_s^n determined by the conditions $\operatorname{Im} s_j(\alpha) = 0,\ j = 2, 3, \dots, n$.

Lemma 1. *If $P(s)$ satisfies conditions 1-3 and has no multiple factors, then for every transformation α, with the exception of a finite number of them,*

$$\operatorname{grad} P(s) \neq 0$$

at the zeros of $P(s)$ situated in the space $R(\alpha)$.

This lemma makes it possible, without loss of generality, to assume that

$$\operatorname{grad} P(s) \neq 0$$

at the zeros of $P(s)$ with real $s_2, s_3, \dots, s_n$.

Let $s_j = \sigma_j + it_j$, $j = 1, 2, \dots, n$. Condition 4 ensures the convexity of the surfaces K_j, $j = 1, 2, \dots, m$. Therefore each K_j, $j = 1, 2, \dots, m$, can be divided into two parts: the "upper" part relative to the σ_1-axis and the "lower" part, i.e., into two such parts that, for fixed $\sigma_2, \sigma_3, \dots, \sigma_n$, at the points of one of them σ_1

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is not less than at the points of the other. We denote one of these parts by l_j^1, the other by l_j^2, $j = 1, 2, \dots, m$. Let $\delta_j = \pm 1$, $j = 1, 2, \dots, m$, where $\delta_j = 1$ if, for fixed $\sigma_2, \sigma_3, \dots, \sigma_n$, σ_1 on l_j^1 is not greater than on l_j^2. If, on the contrary, at the points of l_j^2 σ_1 is not greater than at the corresponding points of l_j^1, then $\delta_j = -1$. Denote

$$\bigcup_j l_j^1 = \Lambda_1$$

and

$$\bigcup_j l_j^2 = \Lambda_2.$$

Depending on how we have divided the set of real zeros of $P(s)$ into the sets Λ_1 and Λ_2, we shall construct different fundamental solutions of equation (1). We now fix one such division.

The set H is constructed in the space $(\sigma_1, \sigma_2, \dots, \sigma_n, \tau_1)$ and consists of the lines $C_{\bar{\sigma}}$. For each $\bar{\sigma} = (\sigma_2, \sigma_3, \dots, \sigma_n)$, $-\infty < \sigma_j < \infty$, $j = 2, 3, \dots, n$, the line $C_{\bar{\sigma}}$ coincides with the straight line parallel to the σ_1-axis and passing through the point $(0, \sigma_2, \sigma_3, \dots, \sigma_n, 0)$, if on this straight line there are no zeros of $P(s)$. If, however, on such a straight line there are zeros of $P(s)$, then we go around them, remaining in the plane $\sigma_j = \text{const}$, $j = 2, 3, \dots, n$, and in such a way that the points of the set Λ_1 remain above (in the direction $\tau_1 > 0$) our line $C_{\bar{\sigma}}$, and the points of the set Λ_2 below our line.

Theorem 1. *If the polynomial $P(s)$ is hypoelliptic, satisfies conditions 1-4, and has no multiple factors, then the integral (2) exists*[1] *and gives a fundamental solution for equation (1).*

If $P(s)$ has factors $Q_j^{\lambda_j}(s)$, $\lambda_j > 1$, then, by condition 3, the polynomials $Q_j(s)$ have no real zeros, and then it is known that the operator

$$\prod_j Q_j^{\lambda_j}\left(i\frac{\partial}{\partial x}\right)$$

has an exponentially decreasing fundamental solution. In this case the fundamental solution for equation (1) is obtained as the convolution of the exponentially decreasing fundamental solution for the operator

$$\prod_j Q_j^{\lambda_j}\left(i\frac{\partial}{\partial x}\right)$$

and the fundamental solution constructed by the method described above for the operator

$$\frac{P}{\prod_j Q_j^{\lambda_j}}\left(i\frac{\partial}{\partial x}\right).$$

[1] The existence of integral (2) may, for example, be understood as follows: for each $\bar{\sigma}$ there exists an integral over $C_{\bar{\sigma}}$ which can be integrated with respect to $\sigma_2, \sigma_3, \dots, \sigma_n$.

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§ 3. Asymptotics of fundamental solutions.

Just as in paper (2), the integral (2) with respect to the variable s_1 is evaluated by means of the theory of residues, and the resulting integrals over the space $(\sigma_2, \sigma_3, \dots, \sigma_n)$ are investigated by means of the saddle-point method (8).

Let

$$r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}, \qquad w_i = x_i/r, \qquad i = 1, 2, \dots, n,$$

$w = (w_1, w_2, \dots, w_n)$. For all w ($w_1 \neq 0$) and any j, $j = 1, 2, \dots, m$, the system

$$w_1 P'_{s_i}(\sigma) - w_i P'_{s_1}(\sigma) = 0, \qquad i = 2, 3, \dots, n,$$

$$P(\sigma) = 0$$

has exactly two real solutions belonging to K_j, one of which belongs to Λ_1, and the other to Λ_2. We shall denote them respectively by

$$\sigma_j^1 = (\sigma_{1j}^1(w), \sigma_{2j}^1(w), \dots, \sigma_{nj}^1(w))$$

and

$$\sigma_j^2 = (\sigma_{1j}^2(w), \sigma_{2j}^2(w), \dots, \sigma_{nj}^2(w)).$$

By ν we shall denote the unit vector of the exterior normal to the surface K_j.

Let $D_p(x) = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ (p_i are nonnegative integers), and let the symbol $|p|$ denote the quantity $p_1 + p_2 + \dots + p_n$.

Theorem 2. *If* $P\left(i\frac{\partial}{\partial x}\right)$ *is a hypoelliptic operator satisfying conditions 1-4, then its fundamental solution constructed above has, as* $r \to \infty$*, the following asymptotic form:*

$$D_p\left(\frac{\partial}{\partial x}\right) E(x) =$$

$$= \sum_{j=1}^{m} \left\{ (2\pi)^{-\frac{n-1}{2}} \frac{\exp\left[-\delta_j \frac{\pi}{4}(n-1)\right]}{\sqrt{K(\sigma_j^\mu)\, \dfrac{\partial P}{\partial \nu}(\sigma_j^\mu)}} D(-i\sigma_j^\mu) \frac{\exp[-i(\sigma_j^\mu, w) r]}{r^{\frac{n-1}{2}}} \right\} + E_p(x),$$

where p *is arbitrary;* $\mu = 1$ *if* $x_1 < 0$*;* $\mu = 2$ *if* $x_1 > 0$*; for* $E_p(x)$ *in a neighborhood of infinity the estimate*

$$|E_p(x)| < \frac{C}{r^{n/2}}.$$

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§ 4. Classes W

Theorem 2 enables us to obtain the classes W for the equations under consideration. Let l be the order of equation (1).

Theorem 3. *If $P\left(i\frac{\partial}{\partial x}\right)$ is a hypoelliptic operator satisfying conditions 1-4, and $f(x)$ is any function having, in a neighborhood of infinity, $[n/2]+l$ derivatives that decrease as $r\to\infty$ faster than $C/r^{n+\varepsilon}$, then there exists a unique solution of equation (1) in the following class of functions W: $u\subset W$, if it is representable as a sum of functions $u(x) =$*

$$= \sum_{j=1}^{m} u_j(x),$$

for which, in a neighborhood of infinity, the inequalities hold:

$$|u_j(x)| < \frac{C}{r^{(n-1)/2}}, \qquad \left|D_p\left(\frac{\partial}{\partial x}\right)u_j(x) - D_p(-i\sigma_j^{\mu})u_j(x)\right| < \frac{C}{r^{n/2}}$$

with any p for which $|p|<l$, where $\mu=1,2$ respectively for $x_1<0$ and $x_1>0$.

Remark. Results analogous to those formulated in the present paper have simultaneously been obtained by V. V. Grushin by another method.

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Note: Figure translations are in progress. See original paper for figures.

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