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## Abstract

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V. I. USHAKOV

# ON A CLASS OF TOPOLOGICAL GROUPS

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In abstract group theory, the so-called finiteness conditions are very popular. In particular, many authors (see, for example, ([3,4])) have studied groups with finite classes of conjugate elements ($FC$-groups). In the present note we consider topological $\overline{FC}$-groups (groups with bicompact classes of conjugate elements) and $\overline{\overline{FC}}$-groups, in which the closures of all classes of conjugate elements are bicompact. The necessity of considering $\overline{\overline{FC}}$-groups alongside $\overline{FC}$-groups follows from the fact that in a topological group the classes of conjugate elements are not always closed. All bicompact and all commutative groups belong to the $FC$-groups. The class of $\overline{FC}$-groups contains locally normal groups, i.e., groups in which every element is contained in a bicompact normal divisor. However, locally normal groups need not be $FC$-groups.

We begin with the consideration of connected simply connected $\overline{FC}$-groups of Lie.

**Lemma.** Let $V$ be an $n$-dimensional complex space, $\Delta \subset E(V)$ an irreducible set of linear transformations of this space; $A \in E(V)$ a linear transformation of the space $V$ with eigenvalues $\lambda_1, \dots, \lambda_n$. If for every $B \in \Delta$ there exists a bicompact set $K_B \subset E(V)$ such that $e^{kA}Be^{-kA} \in K_B$ for every integer $k$, then $\operatorname{Re}\lambda_1 = \dots = \operatorname{Re}\lambda_n$.

**Proof.** Let $e_1, \dots, e_n$ be a basis in which the matrix of the transformation $A$ has Jordan normal form, $A'e_i = \lambda_i e_i$, $i = 1, \dots, n$, $\widetilde{A} = A - A'$. Then $\widetilde{A}^m = 0$ for some $m$. Since $A'$ and $\widetilde{A}$ commute, we have

$$e^{kA}Be^{-kA} = e^{kA'}B_k e^{-kA'}, \qquad B_k = \sum_{0 \le i+j < 2m} \frac{(-1)^j k^{i+j}}{i!j!}\,\widetilde{A}^i B \widetilde{A}^j. \tag{1}$$

Suppose it has already been proved that $\operatorname{Re}\lambda_1 = \dots = \operatorname{Re}\lambda_r$, $n > r \ge 1$, and let $V_r$ be the linear span of the vectors $e_1, \dots, e_r$. In view of the irreducibility of $\Delta$, one may assume that

$$B(e_s) = \sum_{i=1}^{n} b_i e_i, \qquad b_{r+1} \ne 0 \tag{2}$$

for some $B \in \Delta$ and $1 \le s \le r$. Further, by virtue of (1) and (2),

$$e^{kA} B e^{-kA}(e_s) = \sum_{i=1}^{n} b_i(k) e^{(\lambda_i - \lambda_s)k} e_i \in K_B(e_s), \qquad b_i(0) = b_i, \tag{3}$$

where $b_i(k)$, $i = 1, \dots, n$, is a polynomial in $k$ of degree $\le 2m$. Since the set $K_B(e_s)$ is bicompact, all coefficients in (3) are bounded, i.e.,

$$|b_i(k)| e^{\mathrm{Re}(\lambda_i - \lambda_s)k} < c, \qquad 1 \le i \le n,$$

for every integer $k$. Since $b_{r+1}(k)$ is not identically zero ($b_{r+1}(0) = b_{r+1} \ne 0$), $\mathrm{Re}\,\lambda_{r+1} = \mathrm{Re}\,\lambda_s$ ( $b_{r+1}(k)$ will in this case be a constant different from zero).

**Theorem 1.** A connected semisimple $\overline{FC}$-Lie group $G$ is bicompact.

**Proof.** Let $V$ be the Lie algebra of the Lie group $G$, $Z$ the center of $G$, and $\widetilde{G}(V)$ the connected component of the group of automorphisms of the algebra $V$. Then the groups $G/Z$ and $\widetilde{G}(V)$ are isomorphic. Further, $V$ decomposes into a direct sum of simple algebras $V_1, \dots, V_n$. Since $\widetilde{G}(V)$ is the direct product of the groups $\widetilde{G}(V_i)$, $i = 1, \dots, n$, it is enough to prove that if $V$ is a simple Lie algebra and $\widetilde{G}(V)$ is an $\overline{FC}$-group, then $V$ is compact.

It is known that if $V$ is a simple real Lie algebra, then either its complex extension $[V]$ ([2]), §58, $C$ is a simple algebra, or $V$ is obtained from some simple complex algebra $W$ by depriving the latter of complex operators.

In the first case $\widetilde{G}(V)$ can be regarded as an irreducible set of linear transformations of the space $[V]$, for $V$ and $[V]$ have one and the same basis. Since the Lie algebra of the group $\widetilde{G}(V)$ is the adjoint algebra $P$ of the algebra $V$ ([2]), §54, $E$, by the lemma the real parts of all eigenvalues of any endomorphism $p_v$ from $P$ ($v \in V$) are equal to zero. Then the eigenvalues of the transformation $p_v^2$ are nonpositive. Hence, and from the simplicity of $V$, it follows that the scalar square $\mathrm{Sp}\, p_v^2$ is positive definite, i.e. $V$ is compact.

In the second case it is easy to show that $\widetilde{G}(W) = \widetilde{G}(V)$ and $\widetilde{G}(W)$ is irreducible in $W$. Let $s$ be a regular element of the compact form $W'$ of the algebra $W$, and let $A = ip_s$. Since the eigenvalues of $A$ are real ([2]), §62 $A$, by the lemma they are all equal to zero. Hence follows the coincidence of $W'$ with its regular subalgebra, which is commutative, and this contradicts the simplicity of $W$. Thus the second case is in fact impossible.

**Definition.** An element $g$ of a topological group $G$ is called **bicompact** if the closure of the cyclic subgroup generated by $g$ is bicompact. If all elements of the group are bicompact, then the group is called **periodic**; if $G$ has no bicompact elements except the identity, then the group is called **pure**.

**Theorem 2.** A pure locally bicompact normal divisor $H$ of an arbitrary $\overline{FC}$-group $G$ is central.

**Proof.** Let $K$ be the connected component of $H$, and $R$ its solvable radical. In $R$ there is a series of subgroups invariant in $G$, all factors of which are vector groups or tori. The factor group $K/R$ is bicompact by Theorem 1. From Lemma 3.8 of ([6]) it follows that $R = K$ is a vector group. Since $H/K$ is a discrete abelian torsion-free group (see ([5])), $H$ is solvable. If $H$ is commutative, then for any $h \in H$ and $g \in G$

$$h^{kgh^{-k}}g^{-1} = (hgh^{-1}g^{-1})^k,$$

i.e. the subgroup $\{hgh^{-1}g^{-1}\}$ of the group $H$, lying in $S_g \cdot g^{-1}$, is bicompact ($S_g$ is the class of elements of the group $G$ conjugate to $g$). From the purity of $H$ it follows that $hg = gh$. The centrality of an arbitrary pure solvable normal divisor $H$ is proved by induction on the class of solvability of the group $H$.

With the aid of Theorem 2 and Lemma 4 of ([7]), the nilpotency of a connected solvable $\overline{FC}$-group is established. And then, according to the results of V. M. Glushkov (([8]), Theorem 5.1), Lemma 3.8 of ([6]), and Theorem 1, we obtain:

**Theorem 3.** A connected locally bicompact $\overline{FC}$-group is an extension of a bicompact group by means of a vector group.

**Theorem 4.** *In a locally bicompact group, every bicompact invariant set consisting of bicompact elements generates, in the topological sense, a bicompact normal divisor.*

The proof of this theorem, which generalizes the well-known lemma of Dixmier (see ([1]), p. 339), uses Theorem 3 and the lemma from ([5]).

A trivial consequence of Theorem 4 is

**Theorem 5.** *A locally bicompact group is a periodic $\overline{FC}$-group if and only if it is locally normal.*

The main result of the present paper is the following.

**Theorem 6.** *The set of all bicompact elements of a locally bicompact $\overline{FC}$-group $G$ forms a closed characteristic subgroup $P(G)$ (the periodic part of $G$), the quotient group by which is a torsion-free Abelian group.*

**Proof.** The fact that $P(G) = P$ is a subgroup of the group $G$ in the algebraic sense follows from Theorem 4.

We shall prove the closedness of $P$. Let $K$ be the connected component of the group $G$, and let $P'/K$ be the periodic part of the quotient group $G/K$. According to ([5]), $P'$ is an open subgroup of the group $G$. It is not difficult to prove that $P' = PK$. Since $\overline{P}$ is a locally bicompact group, it contains a neighborhood of the identity $U$ with bicompact closure. Consider the subgroup $A = \{\overline{U}\}$ of the group $\overline{P}$. It is open in $\overline{P}$. Using the existence in $A$ of the bicompact set of generators $\overline{U}$ and the everywhere dense set of bicompact elements $A \cap P$, one can show that $A$ is bicompact and, consequently, is contained in $P$. Hence it follows that $P = \overline{P}$.

We shall prove the torsion-freeness of the quotient group $G/P$. The commutativity of $G/P$ will then follow from Theorem 2. Suppose that there is a bicompact element in $G/P$. Then it is contained in $P'$, since, by ([5]), $G/P'$ is a torsion-free group. Therefore it remains to prove the torsion-freeness of the group $PK/P$. Consider the quotient group

$$KP/P(K) = P'/P(K) \cong \widetilde{P}',$$

where $P(K)$ is the periodic part of $K$ (see Theorem 3). Since $\widetilde{P}' = \widetilde{P} \cdot \widetilde{K}$ and $\widetilde{P} \cap \widetilde{K} = 1$, the mapping

$$\varphi : \quad \widetilde{K} \ni \tilde{k} = kP(K) \to \tilde{k}\widetilde{P} \in \widetilde{P}'/\widetilde{P}$$

is a continuous isomorphism (in the algebraic sense) of the group $\widetilde{K}$ onto the whole group $\widetilde{P}'/\widetilde{P}$. By the connectedness and local bicompactness of $\widetilde{K}$, $\varphi$ is a homeomorphism (([2]), Theorem 12). Consequently, $\widetilde{P}'/\widetilde{P} \cong P'/P$ is a torsion-free group.

In the case of discrete groups this theorem becomes the well-known theorem of B. H. Neumann ([4]).

**Definition.** A topological group $G$ is called **bicompactly generated** if it possesses a bicompact system of elements algebraically generating the whole group.

**Theorem 7.** *A locally bicompact group $G$ is a bicompactly generated $\overline{FC}$-group if and only if it is an extension of a bicompact group by means of the direct product of a vector group and a discrete Abelian group without torsion with a finite number of generators.*

**Proof.** It is clear that in the proof only the bicompactness of the periodic part $P$ of the bicompactly generated $\overline{FC}$-group $G$ is needed. If $G$ is totally disconnected, then it contains an open bicompact subgroup $H$, and $G = \{H, x_1, \dots, x_n\}$, where $x_1, \dots, x_n$ belong to a bicompact set of generators of the group $G$. Since $P$ contains the commutator subgroup $G'$ of the group $G$, it is enough to prove the bicompact generation of $G'$, whence the bicompactness of the closure of the commutator subgroup, and hence the bicompactness of $P$, will follow. A bicompact set of generators of the group

$G'$ consists of elements of the following form: 1) $[h, h']$, where $h, h' \in H$; 2) $[s_i^{\varepsilon}, h]$, where $h \in H$, $s_i \in \overline{S}_{x_i}$, $i = 1, \dots, n$; $\varepsilon = \pm 1$; 3) $[s_i^{\varepsilon}, s_j^{\varepsilon'}]$, $s_i \in \overline{S}_{x_i}$, $s_j \in \overline{S}_{x_j}$; $i, j = 1, \dots, n$; $\varepsilon, \varepsilon' = \pm 1$ (here $[x, y]$ denotes the commutator of the elements $x$ and $y$).

If $G$ is an arbitrary locally bicompact group, then consider its connected component $K$. The factor group $KP/K$ is bicompact by what has been proved. From the bicompact generatedness of the group $PK$ it follows that $P/K \cap P \cong KP/K$ is a bicompact group (([2]), § 20, $G$). The bicompactness of $P$ now follows from the bicompactness of $K \cap P$.

**Corollary 1.** *A locally bicompact, bicompactly generated, locally normal group is bicompact.*

**Corollary 2.** *A locally bicompact, bicompactly generated group is an $\overline{FC}$-group if and only if the closure of its commutant is bicompact.*

In conclusion I take this opportunity to express my deep gratitude to L. Ivanovskii for valuable advice.

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