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Soviet-era science, translated into English

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1962

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**Abstract**

**Full Text**

**K. K. Golovkin and V. A. Solonnikov**

## EMBEDDING THEOREMS FOR FRACTIONAL SPACES

*(Presented by Academician V. I. Smirnov, 15 XI 1961)*

In the present paper direct and inverse embedding theorems are given for a broad class of functional spaces that contains a number of previously studied spaces. These theorems are a development of the results of S. M. Nikol'skii<sup>(1)</sup>, O. V. Besov<sup>(2)</sup>, and the authors<sup>(3,4)</sup>. In method the present paper is closely connected with the works of the authors<sup>(3,4)</sup>.

We study spaces of functions whose norm has the form

$$\sum_{i=1}^n \left\{ I \left[ \frac{1}{h^{r_i - r'_i}} \left\| \Delta_{h,i}^{r''_i - r'_i} \frac{\partial^{r'_i} u}{\partial x_i^{r'_i}} \right\|_{p_i, s} \right] + \|u\|_{p_i, n} \right\} \quad (p_i \geq 1). \quad (1)$$

Here  $r_i$  are positive numbers;  $r'_i$  is the greatest integer less than  $r_i$ ;  $r''_i$  is the least integer greater than  $r_i$ ;  $\|\cdot\|_{p, n}$  is the norm in  $L_p$  over  $n$ -dimensional space.  $\|u\|_{p, s}$  is defined as

$$\sup_{x_{s+1}, \dots, x_n} \left\{ \int |u(x_1, \dots, x_s, x_{s+1}, \dots, x_n)|^p dx_1 \dots dx_s \right\}^{1/p}.$$

$I$  is a functional of maximization type (abbreviated f.m.t.). This notion was introduced in<sup>(3)</sup>. Here we formulate another definition, equivalent to the previous one.

We shall say that  $I[\psi(h)]$  is an f.m.t. if it is defined on nonnegative functions  $\psi(h)$  of the positive argument  $h$  by the equality

$$I[\psi(h)] = N[u(x)],$$

where  $u(x) = \psi(e^x)$ ;  $N$  is a norm defined on functions given on the whole line and possessing the following properties: 1)  $N[u(x+\xi)] = N[u(x)]$ ,  $-\infty < \xi < \infty$ ; 2) from the inequality  $u_1(x) \geq u_2(x) \geq 0$  it follows that  $N[u_1(x)] \geq N[u_2(x)]$ .

Examples of f.m.t.'s are

$$I_{\rho,p}[\psi(h)] = \sup_{s>0} \left\{ \int_0^\infty \frac{\rho(hs)\psi^p(h)}{h} dh \right\}^{1/p}, \quad (2)$$

where  $\rho(h)$  is a measurable function satisfying the inequalities  $1 \geq \rho(h) \geq 0$ . In the case when  $\rho(h) \equiv 1$ , we shall denote the functionals (2) simply by  $I_p[\psi(h)]$ .

Other examples of f.m.t.'s may be obtained by taking Orlicz norms for  $N$ . An important subclass of f.m.t.'s is formed by those that are generated by  $N$ -norms invariant with respect to any transformation of  $x$  preserving measure. We shall say that these f.m.t.'s have property  $T$ . The  $I_p$  have property  $T$ . If an f.m.t.  $I$  has property  $T$ , then

$$I[\psi(h^\alpha)] \sim I[\psi(h)] \quad \text{for any } \alpha > 0.$$

The main role in the proof of the direct embedding theorems is played by Theorem 1 of [1] and by the following easily proved lemma.

**Lemma.** If  $I$  has property  $T$ , and  $\|\cdot\|$  is invariant with respect to arbitrary shifts of the arguments of the function  $u(x_1, \dots, x_n)$ , then for sufficiently small  $\delta$  the inequality

$$\sum_{i=1}^n I \left[ \frac{1}{h^{r_i}} \|\Delta_{h,i}^l u\| \right] \leq C \sum_{i=1}^n I \left[ \frac{1}{h^{r_i}} \|\Delta_{h,i}^l u_{\delta h x_1^i, \dots, \delta h x_n^i}\| \right],$$

holds, where  $x_j^i = r_i/r_j$ ;  $u_{\eta_1, \dots, \eta_n}$  is an average of  $u(x_1, \dots, x_n)$  constructed with a kernel having a sufficiently large number of zero moments (beginning with the first) and with parameters  $\eta_1, \dots, \eta_n$ .

Such "high-quality" averages of functions play in our method the same role as approximations by entire functions of finite degree in the method of S. M. Nikol'skii.

Denote by  $H_{J;p_1, \dots, p_n; s}^{r_1, \dots, r_n}$  ( $s \leq n$ ) the closure, in the norm (1), of sufficiently smooth functions defined in the whole space. In the case  $p_1 = p_2 = \dots = p_n = p$  we denote this space by  $H_{J;p; s}^{r_1, \dots, r_n}$ .

We proceed to the formulation of the embedding theorems proved by us.

**Theorem 1.** Let  $u(x) \in H_{J;p_1, \dots, p_n}^{r_1, \dots, r_n}$ ; let  $I$  have property  $T$ , and let the numbers

$$x_i = 1 - \sum_{j=1}^n \left( \frac{1}{p_j} - \frac{1}{p_i} \right) \frac{1}{r_j}$$

be positive. Then, if  $q \geq p_i$  and

$$x = 1 - \sum_{j=1}^n \left( \frac{1}{p_j} - \frac{1}{q} \right) \frac{1}{r_j} > 0,$$

then  $u \in H_{J;q;n}^{r'_1, \dots, r'_n}$ , where  $r'_i = \frac{x r_i}{x_i}$ .

**Remark.** This theorem remains valid if the metrics  $L_{p_i}$  and  $L_q$  are replaced respectively by

$$\sup_{-\infty < z_1, \dots, z_n < \infty} \|v(x+z)\chi_M(x)\|_{L_{p_i}}, \quad \sup_{-\infty < z_1, \dots, z_n < \infty} \|v(x+z)\chi_M(x)\|_{L_p},$$

where  $\chi_M(x)$  is the characteristic function of an arbitrary open set  $M$ .

**Theorem 2.** Let  $u(x) \in H_{J;p;n}^{r_1, \dots, r_n}$ ; let  $I$  have property  $T$ . Then, if

$$x = 1 - \frac{1}{p} \sum_{j=1}^n \frac{1}{r_j} + \frac{1}{q} \sum_{j=1}^n \frac{1}{r_j}$$

lies between 0 and 1 and  $q \geq p$ , then, for  $r'_i = x r_i$ ,

$$u \in H_{I;q;s}^{r_1, \dots, r_n}.$$

**Remark.** If in Theorems 1 and 2  $r_1 = r_2 = \dots = r_n$  and  $p_1 = \dots = p_n$ , then  $I$  need not have property  $T$ .

It seems necessary also to clarify which mixed partial derivatives exist for functions to which Theorems 1 and 2 are applicable.

This question is resolved by the following theorem.

**Theorem 3.** Let

$$I \left[ \frac{1}{h^{r_i}} \|\Delta_{h,i}^{m_i} u\| \right] < \infty,$$

where  $m_i > r_i > 0$ ;  $I$  is an arbitrary f.t.m. having property  $T$ ; the norm  $\|\cdot\|$  is the same as in the lemma. Let the point  $(l_1, \dots, l_n)$  lie in the space  $E_n$  on the same side of the plane intersecting the coordinate axes at the points  $r_i$  as the origin of coordinates. Put

$$s_i = \left( 1 - \sum_{j=1}^n \frac{l_j}{r_j} \right) r_i.$$

Then, if

$$I \left[ \frac{\varepsilon(h)}{h^{s_i+l_1+\dots+l_n}} \left\| \Delta_{h,1}^{l_1} \dots \Delta_{h,n}^{l_n} \Delta_{h,i}^{k_i} u \right\| \right] < \infty,$$

where

$$\varepsilon(h) = \begin{cases} 1, & h \geq 1, \\ 0, & h < 1, \end{cases}$$

and, by assumption,  $k_i > s_i$ , the estimate holds

$$\sum_{i=1}^n I \left[ \frac{1}{h^{s_i}} \left\| \Delta_{h,i}^{k_i} \frac{\partial^{l_1+\dots+l_n} u}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} \right\| \right] \leq C \sum_{i=1}^n I \left[ \frac{1}{h^{r_i}} \left\| \Delta_{h,i}^{m_i} u \right\| \right].$$

**Corollary.** If  $u \in H_{I;p;s}^{r_1, \dots, r_n}$  and

$$\sum_{j=1}^n \frac{l_j}{r_j} < 1,$$

then

$$\partial^{l_1+\dots+l_n} u / \partial x_1^{l_1} \dots \partial x_n^{l_n} \in H_{J;p;s}^{s_1, \dots, s_n},$$

where

$$s_k = r_k \left( 1 - \sum_{j=1}^n \frac{l_j}{r_j} \right).$$

Finally, let us give an embedding theorem for functional spaces that differ only in the functional of maximization type.

**Theorem 4.** If  $J$  is an arbitrary f.t.m., and the norm  $\|\cdot\|$  is the same as in the lemma, then the estimate holds

$$J \left[ \frac{1}{h^{r-r'}} \left\| \Delta_{h,i}^{r''-r'} \frac{\partial^{r'} u}{\partial x_i^{r'}} \right\| \right] \leq CI_1 \left[ \frac{1}{h^{r-r'}} \left\| \Delta_{h,i}^{r''-r'} \frac{\partial^{r'} u}{\partial x_i^{r'}} \right\| \right],$$

$$C = 28I[\chi(h, 1, e)]; \quad \chi(h, 1, e) = \begin{cases} 1, & h \in (1, e), \\ 0, & h \in \bar{(1, e)}. \end{cases}$$

If, in addition,  $I$  has property  $T$ , then

$$I_\infty \left[ \frac{1}{h^{r-r'}} \left\| \Delta_{h,i}^{r''-r'} \frac{\partial^{r'} u}{\partial x_i^{r'}} \right\| \right] \leq CI \left[ \frac{1}{h^{r-r'}} \left\| \Delta_{h,i}^{r''-r'} \frac{\partial^{r'} u}{\partial x_i^{r'}} \right\| \right],$$

$$C = 15I[\chi(h, 1, 2)].$$

**Corollary.** If  $I$  is any f.t.m., then

$$H_{I;p;s}^{r_1, \dots, r_n} \subset H_{I;p;r}^{r_1, \dots, r_n}.$$

If  $I$  is an f.t.m. with properties  $T$ , then

$$H_{I;p;s}^{r_1, \dots, r_n} \subset H_{I_\infty;p;s}^{r_1, \dots, r_n}.$$

We now pass to the formulation of the extension theorem and the inverse embedding theorems.

Let the function  $u(x)$  be given in the half-space  $x_i \geq 0$  and belong there to the space

$$H_{I;p_1, \dots, p_n}^{r_1, \dots, r_n},$$

i.e.,

$$\sum_{j=1}^n \left\{ I \left[ \frac{1}{h^{r_j - r'_j}} \left\| \Delta_{h,j}^{r''_j - r'_j} \frac{\partial^{r'_j} u}{\partial x_j^{r'_j}} \right\|_{L_{p_j}(x_j \geq 0)} \right] + \|u\|_{L_{p_j}(x_j \geq 0)} \right\} < \infty.$$

**Theorem 5.** There exists an extension of the function  $u(x)$  to the whole space  $E_n$  such that

$$\begin{aligned} & \sum_{j=1}^n \left\{ I \left[ \frac{1}{h^{r_j - r'_j}} \left\| \Delta_{h,j}^{r''_j - r'_j} \frac{\partial^{r'_j} u}{\partial x_j^{r'_j}} \right\|_{L_{p_j}(x_j \geq 0)} \right] + \|u\|_{L_{p_j}(x_j \geq 0)} \right\} \leq \\ & \leq C \left\{ \sum_{j=1}^n I \left[ \frac{1}{h^{r_j - r'_j}} \left\| \Delta_{h,j}^{r''_j - r'_j} \frac{\partial^{r'_j} u}{\partial x_j^{r'_j}} \right\|_{L_{p_j}(E_n)} \right] + \|u\|_{L_{p_j}(E_n)} \right\}. \end{aligned}$$

This extension can be constructed by the well-known method of Hestenes and Whitney.

**Theorem 6.** Let  $0 < s < n$ ,  $p > 1$ . Let also a sequence of positive numbers  $l_k$  ( $k = 1, \dots, n$ ) and  $N$  distinct

sequences of positive integers  $\nu_j^{(i)}$  ( $j = s+1, \dots, n$ ;  $i = 1, \dots, N$ ) such that

$$\mu_i = 1 - \sum_{j=s+1}^n \frac{\nu_j^{(i)}}{l_j} - \frac{1}{p} \sum_{j=s+1}^n \frac{1}{l_j} > 0.$$

Then, for the functions  $\varphi^{(i)}(x_1, \dots, x_s) \in H_{l;p;s}^{l_1, i, \dots, l_{s,i}}$  prescribed on the hyperplane  $E_s$  ( $x_{s+1} = \dots = x_n = 0$ ), one can construct a function  $\varphi(x_1, \dots, x_n)$  in the space  $E_n$  such that  $\varphi \in H_{l;p;n}^{l_1, \dots, l_n}$ ,

$$\frac{\partial^{\nu_{s+1}^{(i)} + \dots + \nu_n^{(i)}} \varphi}{\partial x_{s+1}^{\nu_{s+1}^{(i)}} \dots \partial x_n^{\nu_n^{(i)}}} \Big|_{x \in E_s} = \varphi^{(i)}(x_1, \dots, x_s),$$

$$\sum_{j=1}^n I \left[ \frac{1}{h^{l_j}} \|\Delta_{h,j}^m \varphi\|_{L_p(E_n)} \right] + \|\varphi\|_{L_p(E_n)} \leq$$

$$\leq C \sum_{i=1}^N \left\{ \sum_{j=1}^n I \left[ \frac{1}{h^{l_{j,i}}} \|\Delta_{h,j}^m \varphi^{(i)}\|_{L_p(E_s)} \right] + \|\varphi^{(i)}\|_{L_p(E_s)} \right\},$$

where  $l_{j,i} = l_j \mu_i$ , and  $I$  is a functional of maximization type possessing property  $T$ .

Theorem 6 reduces to its particular case  $s = n - 1$ . Further, by virtue of Theorem 5, the function  $\varphi$  can be constructed not in the whole space  $E_n$ , but in the half-space  $x_n \geq 0$ . This construction can be carried out in explicit form, with the function  $\varphi$  expressed as a sum of integral operators of the form

$$\frac{1}{x_n^{\sigma_i}} \int_0^{x_n^{\alpha_1}} \dots \int_0^{x_n^{\alpha_{n-1}}} K_i(y, x_n) \varphi_i(x + y) dy.$$

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Academy of Sciences of the USSR

Received  
18 X 1961

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*Note: Figure translations are in progress. See original paper for figures.*

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