ON THE REPRESENTATION OF OPERATORS OVER MEMORY

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Abstract

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ON THE REPRESENTATION OF OPERATORS OVER MEMORY

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In programming one has to deal with the representation of some operators over memory in the form of a product of other operators.

For what follows we need several definitions given by A. A. Lyapunov in reports at a meeting of the Moscow Mathematical Society and at the Fourth All-Union Mathematical Congress. A **memory** is a set Ω, whose elements are called cells. Let G be a set of elements called the states of cells. A **state of memory** is a mapping $f : \Omega \to G$. Let $\Xi = \{f\}$ be the set of all possible states of memory. An **operator over the memory** Ω (or over Ω) is a mapping $A : \Xi_1 \to \Xi_2$, where Ξ_1 and $\Xi_2 \subset \Xi$. If $A : \Xi_1 \to \Xi_2$ and $B : \Xi_2 \to \Xi_3$, then the **product of the operators** A and B is the operator $C = AB : \Xi_1 \to \Xi_3$, defined as follows: $C(f) = B(A(f))$. Operators over Ω form a category ([1]). A tuple of n cells $x^1, \dots, x^n$, a memory state $f(x)$, and an arbitrary function $\psi(g_1, \dots, g_n)$ of n states determine an n-function of special form: $\varphi(x^1, \dots, x^n / f(x)) = \psi(f(x^1), \dots, f(x^n))$. An n-function of special form is called an (n, m)-operation if $\psi(g_1, \dots, g_n) : G^n \dot{\to} G^m$, i.e. $\varphi(x^1, \dots, x^n / f(x)) = (\bar{g}_1, \dots, \bar{g}_m)$. An ordered collection of m distinct cells $y^1, \dots, y^m$ and an (n, m)-operation determine an (n, m)-operator of special form

$$A(f(x)) = f_1(x) = \begin{cases} \bar{g}_i, & \text{if } x = y^i, \\ f(x), & \text{if } x \neq y^i, \quad i = 1, \dots, m. \end{cases}$$

We consider a memory Ω with an arbitrary set of cells, each of which is capable of being in a finite number of states, and we study the question of representability of operators over the memory Ω in the form of a product of other operators.

Theorem 1. Every (n, m)-operator over Ω is representable in the form of a product of $(n, 1)$-operators over Ω.

Theorem 2. For $m < n$ there exist $(n, 1)$-operators over Ω that are not representable in the form of a product of $(m, 1)$-operators over Ω.

Let the memory Ω' be obtained by adding to the memory Ω one cell taking two states.

Theorem 3. Every (n, m)-operator over Ω is representable in the form of a product of $(2, 1)$-operators over Ω.

The study of more complicated cases requires auxiliary considerations. Let K be an arbitrary category, $\widetilde{K}$ the set of all non-identity elements of the category K, and $\mathfrak{N} \subset \widetilde{K}$. A subcategory $K_1 \subset K$ will be called **regular** if every identity element of the subcategory K_1 is an identity element of the category K. The smallest regular subcategory $K(\mathfrak{N})$ such that $K(\mathfrak{N}) \supset \mathfrak{N}$ will be called the **categorical closure** of the set $\mathfrak{N}$. Sets $\mathfrak{N}_1$ and $\mathfrak{N}_2 \subset \widetilde{K}$ are called **equivalent** if $K(\mathfrak{N}_1) = K(\mathfrak{N}_2)$. The question of the possibility of representing an operator A in the form of a product of operators from the set $\mathfrak{N}$ reduces to the question of the equivalence of the sets $\mathfrak{N}$ and $\mathfrak{N} \cup \{A\}$.

We describe a certain class of criteria for the equivalence of subsets of an arbitrary category K. Let K_α be some operator category defined on the set Ξ_α. On the set Ξ_α define a subordination relation $\overset{r}{<}$, satisfying the following axioms: 1) if $\xi_1 \overset{r}{<} \xi_2$ and $\xi_2 \overset{r}{<} \xi_3$, then $\xi_1 \overset{r}{<} \xi_3$; 2) if $\xi_1 \overset{r}{<} \xi_2$, then $A^\alpha \xi_1 \overset{r}{<} A^\alpha \xi_2$ for every operator $A^\alpha \in K_\alpha$ (provided that the operator A^α is defined on ξ_1 and ξ_2). By a **quasi-invariant** of a set $\mathfrak{M} \subset K_\alpha$ we shall mean any element $\xi \in \Xi_\alpha$ such that, for every A^α from $\mathfrak{M}$, one has $A^\alpha \xi \overset{r}{<} \xi$. The totality of all quasi-invariants of the set $\mathfrak{M}$ is denoted by $S(\mathfrak{M})$. Let $\sigma : K \to K_\alpha$ be a homomorphism of the category K into the category K_α.

If $\mathfrak{N} \subset \widetilde{K}$ and $R = (K_\alpha, \overset{r}{<}, \sigma)$, then by the R**-characteristic** of the set $\mathfrak{N}$ we shall mean the set $R(\mathfrak{N}) = S(\sigma\mathfrak{N})$. The sets $\mathfrak{N}_1$ and $\mathfrak{N}_2 \subset \widetilde{K}$ are called R**-equivalent** if $R(\mathfrak{N}_1) = R(\mathfrak{N}_2)$.

An R**-criterion of equivalence**. In order that the sets $\mathfrak{N}_1$ and $\mathfrak{N}_2 \subset \widetilde{K}$ be equivalent, it is necessary that they be R-equivalent.

Thus, to each triple $(K_\alpha, \overset{r}{<}, \sigma) = R_\beta$ there corresponds an R_β-criterion of equivalence. We shall say that the R_1-criterion is **weaker** than the R_2-criterion (the R_2-criterion is stronger than the R_1-criterion) and write $R_1 \preccurlyeq R_2$, if from the R_2-equivalence of arbitrary sets $\mathfrak{N}_1$ and $\mathfrak{N}_2 \subset \widetilde{K}$ there follows their R_1-equivalence. The R_1-criterion and the R_2-criterion are called **equally strong** if $R_1 \preccurlyeq R_2$ and $R_2 \preccurlyeq R_1$. Identifying equivalent R-criteria, we obtain a partially ordered set $\mathfrak{R}(K)$, whose elements are classes of equally strong R-criteria of equivalence.

Theorem 4. *The partially ordered set* $\mathfrak{R}(K)$ *is a complete lattice.*

Theorem 5. *The unit element of the lattice* $\mathfrak{R}(K)$ *is the class of sufficient* R*-criteria of equivalence.*

A set $H \subset \widetilde{K}$ is called $\widetilde{K}$-closed if $\widetilde{K} \cap K(H) \subset H$. A system $\mathcal{E}$ of nonempty $\widetilde{K}$-closed subsets of the set $\widetilde{K}$ is called **proper** if: 1) $\widetilde{K} \in \mathcal{E}$ and 2) together with the sets $H_\alpha \in \mathcal{E}$ $(\alpha \in I)$, the system $\mathcal{E}$ also contains their intersection $\bigcap_{\alpha \in I} H_\alpha$, if it is nonempty.

If $\mathcal{E}$ is a proper system and $N \subset \widetilde{K}$, then by $\mathcal{E}N$ we shall denote the smallest set $H \in \mathcal{E}$ containing the set N.

Every R-criterion of equivalence generates a certain partition D_R of the system of all subsets of the set $\widetilde{K}$ into classes of R-equivalent sets. To each class Q_α of the partition D_R we put in correspondence the set

$$H_\alpha = \bigcup_{N \in Q_\alpha} N.$$

The totality of the sets H_α corresponding to all possible classes of the partition D_R will be denoted by $\mathcal{E}_R$. Thus, to each R-criterion of equivalence there corresponds a certain system of sets $\mathcal{E}_R$.

Theorem 6. *The system of sets $\mathcal{E}_R$ corresponding to an R-criterion of equivalence is proper.*

The question arises: does every proper system correspond to some R-criterion of equivalence?

Let $\overline{S}$ be a semigroup with a system of generators K and a system of defining relations $\mathfrak{A}$, consisting of all relations valid in the category K. In what follows we shall regard the category K as a subset of the semigroup $\overline{S}$, and by NM, where $N, M \subset \overline{S}$, we shall mean the product of sets in the semigroup $\overline{S}$. If $N \subset K$ and

$\dot{T}(N) = \{a : aN \cap K \neq \Lambda,\ a \in K\}$, then by $F(N)$ we denote the set $\{a : T(N)a \cap K \neq \Lambda,\ a \in K\}$.

Let $\mathcal{E}$ be a regular system and $H \in \mathcal{E}$. A set H is called **distinguishable** in $\mathcal{E}$ if there exist a set $S \subset \overline{S}$ and a mapping $f(a) = P_a$ $(a \in S,\ P_a \subset S)$ satisfying the following conditions:

1°. $F(H) \subset S$.

2°. If $ab \in S$ and $b \in K$, then $a(bK \cap K) \subset S$.

3°. If $a \in F(H)$ and $b \in H$, then $S \cap H_a \subset P_a$ and $S \cap P_{ba} \subset P_a$.

4°. If $aN \subset P_a$ $(a \in S,\ N \subset \widetilde{K})$, then $a\mathcal{E}N \subset P_a$ and $aF(\mathcal{E}N) \subset S$.

5°. If $a \in P_b$, then $P_a \subset P_b$.

6°. If $a \in P_b,\ c \in K$ and $ac,\ bc \in S$, then $ac \in P_{bc}$.

7°. If $a \in H$, then $\widetilde{K} \cap P_a \subset H$.

Otherwise the set H is called **indistinguishable** in $\mathcal{E}$.

Theorem 7. *In order that there exist an R-equivalence criterion corresponding to the regular system $\mathcal{E}$, it is necessary and sufficient that every set $H \in \mathcal{E}$ be distinguishable in $\mathcal{E}$.*

Theorem 8. *If there exist elements $a, b \in K$ and a set $N \subset \widetilde{K}$ such that $ab \in H$, $aN \subset Ha \cap K$, and $a\mathcal{E}Nb \cap (\widetilde{K}/H) \neq \Lambda$, then the set $H \in \mathcal{E}$ is indistinguishable in $\mathcal{E}$.*

The following theorem gives sufficient conditions for distinguishability under the assumption that the category K is a semigroup.

Theorem 9. *If for arbitrary $a \in K$ and $N \subset \widetilde{K}$, from $aN \subset Ha \cup a$ it follows that $a\mathcal{E}N \subset Ha \cup a$, then the set $H \in \mathcal{E}$ is distinguishable in $\mathcal{E}$.*

Let us apply the results obtained to groups. Let K be an arbitrary group, and let H be an arbitrary $\widetilde{K}$-closed set of nonidentity elements of the group K.

Theorem 10. *In order that the set H be distinguishable in $\mathcal{E}_0 = \{\widetilde{K}, H\}$, it is necessary and sufficient that for every $a \in K$ the relation $aH \subset Ha$ hold.*

Theorem 11. *In order that the subgroup K_1 be a normal divisor of the group K, it is necessary and sufficient that there exist an R_0-equivalence criterion corresponding to the system $\mathcal{E}_0 = \{\widetilde{K}, K_1 \cap \widetilde{K}\}$.*

Theorem 12. *In order that to every regular system there correspond some R-equivalence criterion, it is necessary and sufficient that the group K be abelian or Hamiltonian.*

As an illustration, let us give one example. Let the memory $\Omega = \{x_1, x_2, x_3\}$ consist of three cells, each of which can be in two states (0 and 1). $\Xi = \{f(x)\}$ is the set of all states of the memory Ω. The category K consists of all invertible operators $A : \Xi \to \Xi$; $\mathfrak{N}$ is the set of all $(2,1)$-operators from K. We construct an $\widetilde{R}$-criterion recognizing the equivalence of the sets $\mathfrak{N}$ and $\mathfrak{N} \cup \{A\}$, where A is an arbitrary operator from K. Let $E = \{e\}$ be the set whose elements are all four-element subsets $e \subset \Xi$, and let $\Xi_1 = \{\xi\}$ be the set of all subsets $\xi \subset E$ consisting of 14 elements of the set E. The operator category K_1 consists of all invertible operators $A_1 : \Xi_1 \to \Xi_1$.

The subordination relation $\overset{1}{<}$ in Ξ_1 is defined as follows: $\xi_1 \overset{1}{<} \xi_2$ if and only if $\xi_1 = \xi_2$. As the homomorphism $\sigma_1 : K \to K_1$ we take the mapping which assigns to an operator $A \in K$ the corresponding operator $A^1 \in K_1$ induced by the operator A.

If $\widetilde{R} = (K_1, \overset{1}{<}, \sigma_1)$, then the $\widetilde{R}$-criterion completely solves the question of the equivalence of the sets $\mathfrak{N}$ and $\mathfrak{N} \cup \{A\}$: in order that the sets $\mathfrak{N}$ and $\mathfrak{N} \cup \{A\}$ be equivalent, it is necessary and sufficient that they be $\widetilde{R}$-equivalent. The $\widetilde{R}$-equivalence of $\mathfrak{N}$ and $\mathfrak{N} \cup \{A\}$ is equivalent to the relation $\widetilde{R}(\mathfrak{N}) \subset \widetilde{R}(A)$. In the present case the $\widetilde{R}$-characteristic of the set

$\mathfrak{N}$ consists of a single element $\xi_0 = \{e_1, \dots, e_{14}\}$, where the sets $e_i \subset \Xi$ $(i = 1, \dots, 14)$ are characterized by the following property: $\sum_{f \in e_i} f(x_j) = r_j^i$, $(i = 1, \dots, 14;\ j = 1, 2, 3)$, is an even number. Thus, in order that an operator $A \in K$

be representable as a product of $(2,1)$-operators over Ω, it is necessary and sufficient that $\xi_0 \in \widetilde{R}(A)$.

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References

1. A. G. Kurosh, A. Kh. Livshits, E. G. Shulgeifer, *UMN*, **15**, no. 6 (96) (1960).

Note: Figure translations are in progress. See original paper for figures.