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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

M. G. SHUR

## ON THE MARTIN BOUNDARY FOR LINEAR ELLIPTIC OPERATORS OF SECOND ORDER

*(Presented by Academician A. N. Kolmogorov on 2 I 1962)*

In Martin's paper <sup>(5)</sup> the first boundary-value problem was studied for the Laplace operator in an arbitrary domain  $D$  of  $m$ -dimensional Euclidean space  $R_m$ . Martin singled out, in the totality of all functions harmonic in  $D$ , the class of minimal positive functions and showed that any nonnegative harmonic function in  $D$  can be represented as an integral over the set of minimal functions. This work of Martin served as the starting point for many investigations. In most of these works, only the case of the Laplace operator was considered, and only recently have works of a more general character begun to appear <sup>(1)</sup>.

In recent years it has become clear that a number of problems connected with Martin theory admit a probabilistic interpretation <sup>(2,4)</sup>, and this circumstance enabled Doob and Hunt to construct Martin boundaries for Markov chains. In the case of a Markov process with an arbitrary phase space, in particular in the case of a diffusion process in some domain with an irregular boundary, the methods of Hunt and Doob are inapplicable.

In the present paper the Martin boundary is constructed for the linear elliptic operator

$$L = \sum_{i,j=1}^m a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i(x) \frac{\partial}{\partial x_i}, \quad x = (x_1, \dots, x_m),$$

given in a domain  $D$  ( $D \subseteq R_m$ ) and nondegenerate inside  $D^*$ . It is assumed that the functions  $a_{ij}$  have continuous second derivatives in  $D$ , and the functions  $b_i$  have continuous first derivatives in  $D$ , all these derivatives satisfying a Hölder condition with exponent  $\lambda > 0$  in any bounded domain contained together with its closure in  $D$ .

1. To the operator  $L$  we associate a strict Markov process  $X = (x_t, \tau, \mathcal{M}_t, P_x)$ , which is killed at the moment of first exit from the domain  $D$ , and whose infinitesimal operator  $\mathfrak{A}$  (in the sense of E. B. Dynkin) coincides with  $L$  on the set of functions twice continuously differentiable in  $D^{**}$ . Recall that the process  $X$  is called recurrent if in the domain  $D$  there exists a subdomain  $U$  with compact closure contained in  $D$  such that, for every

$x \in D$ , the trajectory  $x_t$ ,  $P_x$ -almost surely, reaches  $U$  before the time  $\tau$ . Analytic criteria for recurrence of the process in the case  $D = R_m$  can be found in (7).

**Theorem 1.** *In order that the equation  $Lu = 0$  have in the domain  $D$  at least one nonnegative fundamental solution  $v(x, y)$ , it is necessary and sufficient that the process  $X$  be nonrecurrent.*

A result close to Theorem 1 can be found in (3). If the process  $X$  is nonrecurrent, then the fundamental solution referred to in Theorem 1 can be constructed as follows. Choose a nondecreasing sequence of bounded domains  $F_n$ , each of which belongs to the class  $A^{(1, \lambda_n)}$

\* On the boundary of the domain  $D$  the operator  $L$  may have a degeneration of any kind.

\*\* For terminology see (6,9).

for some  $\lambda_n > 0$ , and let  $\bigcup_n F_n = D$ . Then for every  $n$  there exists a Green's function  $v_n(x, y)$  for the Dirichlet problem  $Lu = f$  in the domain  $F_n$  (11). It is not hard to show that the sequence  $v_n(x, y)$  is nondecreasing and that the function

$$v(x, y) = \lim_{n \rightarrow \infty} v_n(x, y) \quad (1)$$

is a nonnegative fundamental solution of the equation  $Lu = 0$  in the domain  $D$ , if the process  $X$  is transient.

2. Let us now consider the cone of nonnegative solutions of the equation  $Lu = 0$  in the domain  $D$ . In the case of a recurrent process  $X$ , this cone consists of constants alone. Therefore, in what follows we shall assume that the process  $X$  is transient.

On the set  $D \times D$  define the kernel  $K(x, y)$ :

$$K(x, y) = \begin{cases} \frac{v(x, y)}{v(x_0, y)}, & \text{if } y \neq x_0, \\ 0, & \text{if } y = x_0, x \neq x_0, \\ 1, & \text{if } x = y = x_0, \end{cases}$$

where  $x_0$  is some fixed point of  $D$ , and the function  $v(x, y)$  is given by formula (1). Following Martin, we shall call a sequence of points  $y_n \in D$  fundamental if it has no limit points inside  $D$  and if, as  $n \rightarrow \infty$ , the functions  $K(x, y_n)$  tend to some function  $g(x)$  that is a solution of the equation  $Lu = 0$  in the domain  $D$ . Two fundamental sequences are called equivalent if they correspond to one and the same solution  $g(x)$  of the equation  $Lu = 0$ . The set of fundamental sequences is divided into classes of mutually equivalent sequences. The totality of these classes forms the Martin boundary  $\Delta$ .

Put  $E = D \cup \Delta$  and extend  $K(x, y)$  to all of  $D \times E$ , setting  $K(x, y) = g(x)$  for  $y \in \Delta$  (here  $g(x)$  is the solution of the equation  $Lu = 0$  corresponding to the point  $y \in \Delta$ ). Then introduce in  $E$  the metric  $\rho(x, y)$ :

$$\rho(x, y) = \int_G \frac{|K(z, x) - K(z, y)|}{1 + |K(z, x) - K(z, y)|} dz,$$

where  $G$  is a fixed, once and for all, bounded subdomain of  $D$ , contained in  $D$  together with its closure, and  $dz$  is the element of Lebesgue measure in  $R_m$ . Repeating the arguments of <sup>(5)</sup>, we establish that  $E$  is compact in the metric  $\rho$  and that the topology induced by  $\rho$  in  $D$  coincides with the ordinary topology in  $R_m$ . Relying on the results of <sup>(8)</sup>, one can show that any nonnegative solution  $u(x)$  of the equation  $Lu = 0$  in the domain  $D$  is representable in the form

$$u(x) = \int_{\Delta} K(x, y) \mu(dy),$$

where  $\mu$  is some Borel measure in  $E$ . Hence it follows immediately that all minimal solutions of the equation  $Lu = 0$  in the domain  $D^*$  have the form  $aK(x, y)$ , where  $y \in \Delta$ ,  $a = \text{const}$ . Denote by  $\Delta_1$  the totality of all minimal solutions of the equation  $Lu = 0$  in the domain  $D$ . Following Martin in the main, we arrive at theorem 2.

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\* A nonnegative solution  $u(x)$  of the equation  $Lu = 0$  in the domain  $D$  is called minimal if every nonnegative solution of this equation in  $D$  not exceeding  $u(x)$  differs from  $u(x)$  only by a constant factor.

**Theorem 2.** Every nonnegative solution  $u(x)$  of the equation  $Lu = 0$  in the domain  $D$  is representable (and, moreover, in a unique way) in the form

$$u(x) = \int_{\Delta_1} K(x, y) \mu_1(dy), \quad x \in D, \quad (2)$$

where  $\mu_1$  is a certain Borel measure concentrated on  $\Delta_1$ .

The existence and uniqueness of an integral decomposition of a nonnegative solution of the equation  $Lu = 0$  into minimal solutions also follows, as Brelot showed <sup>(1)</sup>, from certain results of Choquet. However, it does not follow from these results that this decomposition has the form (2). Decomposition (2) can be used to find the Martin boundary in concrete cases (see, for example, <sup>(11)</sup>).

Theorem 2 is a special case of the following result.

**Theorem 3.** Every nonnegative  $L$ -superharmonic\*  $u(x)$  in  $D$  is representable (and, moreover, in a unique way) in the form

$$u(x) = \int_{\Delta_1} K(x, y) \mu_1(dy) + \int_D v(x, y) \mu_2(dy), \quad (3)$$

where  $\mu_1$  and  $\mu_2$  are Borel measures concentrated, respectively, on  $\Delta_1$  and in  $D$ .

3. In conclusion, let us dwell on the probabilistic properties of the Martin boundary.

**Theorem 4.** As  $t \uparrow \tau(\omega)$ , the trajectory  $x_t(\omega)$ ,  $P_x$ -almost surely ( $x \in D$ ), converges in the metric  $\rho$  to some limit  $\xi(\omega)$ , with  $\xi(\omega) \in \Delta_1$ .

Let the decomposition (2) for  $u(x) \equiv 1$  have the form

$$1 = \int_{\Delta_1} K(x, y) \mu_0(dy).$$

The measure  $\mu_0$  admits a simple probabilistic interpretation:

$$\mu_0(A) = P_{x_0} \{ \xi(\omega) \in A \}.$$

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\* A function  $f(x)$  lower semicontinuous in  $D$  is called  $L$ -superharmonic in  $D$  if it has the following property: whatever the domain  $D_1 \in A^{(1, \delta)}$  ( $\delta > 0$ ), contained in  $D$  together with its closure, and whatever the solution  $g(x)$  of the equation

$Lu = 0$ , continuous in the closure of  $D_1$  and not exceeding  $f(x)$  on the boundary of the domain  $D_1$ , everywhere in  $D$  the function  $f(x)$  is not less than  $g(x)$ .

*Note: Figure translations are in progress. See original paper for figures.*

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