# MATHEMATICS

V. P. GROMOV

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## Abstract

## Full Text

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V. P. GROMOV

# ON SERIES WITH RESPECT TO THE SYSTEM $\{f(\lambda_n z)\}$

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In recent years, in many works, sequences of linear aggregates formed from functions $f(\lambda_n z)$ have been studied from various points of view, where $f(z)$ is an entire function of finite order with nonzero Taylor coefficients, and $\{\lambda_n\}$ is a sequence of complex numbers with finite convergence exponent. In particular, in the work ([1]) it was shown that, if

$$\lim_{n\to\infty} \frac{n}{|\lambda_n|^\rho} = \tau < \infty$$

and the sequence

$$\left\{ P_n(z) = \sum_{j=1}^{p_n} d_{nj} f(\lambda_j z) \right\}$$

converges uniformly in some sufficiently large disk with center at the origin, then its limiting function $F(z)$ satisfies a certain functional equation, the limits

$$\lim_{n\to\infty} d_{nj} = d_j \quad (j = 1, 2, ...)$$

exist, and two sequences of the indicated form converge to one and the same function if and only if the limits of the corresponding coefficients are equal. Thus to each limiting function $F(z)$ there corresponds a unique series

$$\sum_{j=1}^{\infty} d_j f(\lambda_j z), \tag{1}$$

which, for some $\{\lambda_n\}$ and some functions $F(z)$, may (see ([1]), p. 59) diverge, even everywhere. In the present article we shall indicate some new results concerning series of the form (1).

Let first

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n/\rho + 1)}, \qquad \rho > \frac{1}{2}. \tag{2}$$

Everywhere in what follows

$$\lim_{n\to\infty} \frac{\ln n}{|\lambda_n|^\rho} = 0.$$

Suppose that the series $\sum_{n=1}^{\infty} d_n$ converges. Let $\psi \in [0, 2\pi]$ and let $a > 0$ be sufficiently small; put

$$k(\psi, a) = \lim_{k\to\infty} \frac{\ln |d_{n_k}|}{|\lambda_{n_k}|^\rho},$$

where

$$\lambda_{n_k} = |\lambda_{n_k}| e^{i\varphi_{n_k}} \quad (k = 1, 2, ...)$$

are those numbers of the sequence $\{\lambda_n\}$ for which

$$\psi - a \le \varphi_{n_k} < \psi + a$$

(in the case when the angle $\psi - a \le \varphi < \psi + a$ contains only finitely many $\lambda_n$, we shall put $k(\psi, a) = +\infty$). Let further

$$k(\psi) = \lim_{a\to 0} k(\psi, a).$$

**Theorem 1.** The series (1), in the case (2), converges absolutely in the domain $D$ (uniformly in every finite closed part of $D$), whose points $z = re^{i\varphi}$ satisfy the condition

$$r^\rho \cos^+ \rho(\psi + \varphi) - k(\psi) < 0$$

for every $\psi$, and diverges outside $\overline{D}$. The domain $D$ is open and star-shaped with respect to the origin; in the case $1/2 < \rho \le 1$ it is, moreover, convex.

We note that for $\rho = 1$ the function (2) is $e^z$, and the series (1) becomes a Dirichlet series. For this case a similar theorem was obtained in the work ([3]).

**Theorem 2.** Let $f(z) = \sum_0^\infty a_n z^n$, and suppose that for the function

$$\gamma(t) = \sum_{n=0}^{\infty} \frac{a_n \Gamma(n/\rho + 1)}{t^{n+1}}$$

all singular points lie on the segment $[0, 1]$ of the real axis. Then the series (1) converges absolutely in the domain $D$ (the domain $D$ is defined in Theorem 1). In every bounded closed part of the domain $D$ this series converges uniformly.

In what follows, by $f(z)$ we shall mean an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

of order $\rho$ and type 1, for which

$$a_n \ne 0 \quad (n = 0, 1, 2, ...), \qquad \lim_{n\to\infty} n^{1/\rho} \sqrt[n]{|a_n|} = (e\rho)^{1/\rho}. \tag{3}$$

**Theorem 3.** Let $f(z)$ be an entire function of order $\rho > 1/2$, of type 1, with property (3). Then the radius $R$ of the maximal circle with center at the origin, inside which the series

$$\sum_1^\infty d_n f(\lambda_n z)$$

converges, is determined by the formula

$$R = (k)^{1/\rho}, \qquad k = -\overline{\lim_{n\to\infty}} \frac{\ln|d_n|}{|\lambda_n|^\rho}.$$

Consider the sequence

$$P_n(z) = \sum_{j=1}^{p_n} d_{nj} f(\lambda_j z) \quad (n = 1, 2, \ldots), \qquad \overline{\lim_{n\to\infty}} \frac{n}{|\lambda_n|^\rho} = \tau < \infty, \tag{4}$$

where $d_{nj}$ are certain complex numbers.

**Theorem 4.** Let the sequence (4) converge uniformly in the disk

$$|z| < R, \qquad R > \left(\frac{\pi\tau}{\sin \pi\rho/m}\right)^{1/\rho} = \mu \quad (m \text{ is an integer } > \rho)$$

to an entire function $F(z)$ of order $\nu > \rho$, and let the sequence $\{\lambda_n\}$ satisfy the additional condition

$$\ln\left|\frac{1}{L'(\lambda_n)}\right| = o\left(|\lambda_n|^{\frac{\nu\rho-\varepsilon}{\nu-\rho}}\right), \qquad \varepsilon > 0, \qquad L(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z^m}{\lambda_i^m}\right). \tag{5}$$

Then the series

$$\sum_{j=1}^{\infty} d_j f(\lambda_j z), \qquad d_j = \lim_{n\to\infty} d_{nj},$$

converges to $F(z)$ in the whole plane and the relation

$$\overline{\lim_{n\to\infty}} \frac{\ln|\lambda_n|^\rho}{\ln\ln\dfrac{1}{|d_n|}} = \frac{\nu-\rho}{\nu} \tag{6}$$

holds.

Under the condition that $F(z)$ is an entire function of order $\nu > \rho$ and type $\sigma$, along with (6) another relation also holds:

$$\overline{\lim_{n\to\infty}} \frac{(|\lambda_n|^\rho)^{\frac{\nu}{\nu-\rho}}}{\left(\ln\dfrac{1}{|d_n|}\right)^{\frac{\nu}{\nu-\rho}}} = \left(\frac{\sigma\nu}{\rho}\right)^{\frac{\rho}{\nu-\rho}}. \tag{7}$$

Theorem 4 supplements the following known assertion (see ([1]), p. 54):

If the sequence (4) converges uniformly in the disk $|z| < R,\ R > \mu$, to an entire function $F(z)$ and

$$\delta = \overline{\lim_{n\to\infty}} \frac{1}{|\lambda_n|^\rho} \ln\left|\frac{1}{L'(\lambda_n)}\right| < \infty, \tag{8}$$

then the series

$$\sum_{n=1}^{\infty} d_n f(\lambda_n z)$$

converges to $F(z)$ in the whole plane.

Condition (5) for $\nu > \rho$ is weaker than the condition

$$\ln\left|\frac{1}{L'(\lambda_n)}\right| = o\left(|\lambda_n|^\rho \ln|\lambda_n|\right), \tag{9}$$

and this latter is weaker than condition (8). Condition (8) is satisfied, in particular, if

$$\lim_{n\to\infty}\left(|\lambda_{n+1}|^\rho - |\lambda_n|^\rho\right) = h > 0.$$

In the case when the order of the function $F(z)$ is unknown, but the coefficients of the series are known, the following theorem may be used to compute the order:

**Theorem 5.** Suppose

$$\overline{\lim_{n\to\infty}} \frac{n}{|\lambda_n|^\rho} = \tau < \infty$$

and condition (9) is satisfied.

Then, in order that the series (1) converge in the entire plane and represent an entire function $F(z)$ of finite order $\nu$, it is necessary and sufficient that relation (6) hold. In the case $\nu > \rho$, the type $\sigma$ of the function $F(z)$ is computed by formula (7).

Let $F(z)$ be an entire function; put

$$M(r) = \max_{|z|=r}|F(z)|.$$

The quantity

$$\chi = \overline{\lim_{r\to\infty}} \frac{\ln\ln M(r)}{r^\rho}$$

will henceforth be called the $A$-order of the function $F(z)$.

**Theorem 6.** Suppose the sequence (4) converges uniformly in the disk $|z| < R$, $R > \mu$, to an entire function $F(z)$ of finite $A$-order $\chi$. If the sequence $\{\lambda_n\}$,

$$\overline{\lim_{n\to\infty}} \frac{n}{|\lambda_n|^\rho} = \tau < \infty,$$

satisfies condition (9), then the series

$$\sum_{n=1}^{\infty} d_n f(\lambda_n z),$$

corresponding to the function $F(z)$, converges in the entire plane to $F(z)$, and the relation

$$\overline{\lim_{n\to\infty}} \frac{|\lambda_n|^\rho \ln |\lambda_n|^\rho}{\ln \dfrac{1}{|d_n|}} = \chi \tag{10}$$

holds.

The following assertion is also true:

If (10) holds, then the series (1) converges in the entire plane and represents an entire function of finite $A$-order $\chi$.

For $f(z) = e^z$ ($\rho = 1$), the series

$$\sum_{n=1}^{\infty} d_n f(\lambda_n z)$$

becomes the Dirichlet series

$$\sum_{1}^{\infty} d_n e^{\lambda_n z}.$$

For a Dirichlet series, under conditions on $\lambda_n$ more stringent than ours, formula (10) was established by Valiron ([4]). In the case $\lambda_n > 0$,

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0,$$

Ritt, for the function

$$F(s) = \sum d_n e^{\lambda_n s} \quad (s = \sigma + it),$$

introduced the quantity

$$\chi = \varlimsup_{\sigma\to+\infty} \frac{\ln\ln M(\sigma)}{\sigma}, \qquad M(\sigma) = \sup_{|t|<\infty} |F(\sigma+it)|,$$

which is now called the $R$-order of the function $F(s)$. It has been proved (see, for example, ([5])) that the $R$-order $\chi$ of the function $F(s)$ is computed by formula (10) (where one must put $\rho = 1$).

A. O. Gelfond ([2]) established a connection between the density of the set of zero Taylor coefficients of an entire periodic function and its growth. It turns out that an entire periodic function cannot have very many zero Taylor coefficients. We show that the same property is possessed by an entire function represented in the entire plane by a series of the form (1).

**Theorem 7.** Let

$$f(z) = \sum_0^\infty a_n z^n, \qquad a_n \neq 0 \quad (n = 1, 2, ...),$$

be an entire function of order $\rho > 1/2$, of type 1; let $\lambda_n$ $(n = 1, 2, ...)$ be positive numbers, with

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n^\rho} = 0.$$

Suppose that the coefficients $d_n$ of series (1) satisfy the condition

$$\varlimsup_{n\to\infty} \frac{\ln|d_n|}{\lambda_n^\rho} = -\infty$$

(this condition ensures the convergence of series (1) in the whole plane). If $n_k$ are the indices of the nonzero Taylor coefficients of the function $F(z)$—the sum of series (1)—then

$$\varlimsup_{k\to\infty} \frac{k}{n_k} \geq 1 - \frac{1}{2\rho}.$$

For entire functions of finite order, a more precise theorem holds:

**Theorem 8.** If, under the conditions of the preceding theorem, $F(z)$ is an entire function of finite order $\nu > \rho$, then

$$\varlimsup_{k\to\infty} \frac{k}{n_k} \geq 1 - \frac{\nu-\rho}{2\nu\rho}.$$

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Moscow Power Engineering
Institute

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*Note: Figure translations are in progress. See original paper for figures.*