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**Abstract**

**Full Text**

**MATHEMATICS**

**E. S. GOLOD**

**ON THE HOMOLOGIES OF SOME LOCAL RINGS**

*(Presented by Academician L. S. Pontryagin on 10 I 1962)*

Let  $A$  be a local ring with maximal ideal  $\mathfrak{m}$  (here and below all local rings under consideration are assumed to be commutative Noetherian rings). Let  $x_1, \dots, x_m$  be a minimal system of generators of the ideal  $\mathfrak{m}$ . The Koszul complex of the ring  $A$  is the exterior algebra  $K = \Lambda A^m$  of the free  $A$ -module  $A^m$  with generators  $e_1, \dots, e_m$ , which is regarded as a differential algebra with differential defined by the following formula (cf. (2)):

$$d(e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{\alpha=1}^k (-1)^{\alpha-1} x_{i_\alpha} e_{i_1} \wedge \dots \wedge \hat{e}_{i_\alpha} \wedge \dots \wedge e_{i_k}.$$

Some special cases are known in which information about the homologies of the Koszul complex of the ring  $A$  makes it possible to obtain information about the homologies of the ring  $A$  itself. If the Koszul complex is acyclic (or even if it is known only that  $H_1(K) = 0$ ), then the homology algebra of the ring  $A$  is an exterior algebra with  $m$  generators of dimension 1, and the ring  $A$  itself is a regular local ring (see (2,5)). If the homology algebra of the Koszul complex is an exterior algebra over  $H_1(K)$  (or even if only  $[H_1(K)]^2 = H_2(K)$ ), then the homology algebra of the ring  $A$  has only generators of dimensions 1 and 2, and the ring  $A$  itself is a local complete intersection (1). Finally, let the dimension of the group  $H_i(K)$  be  $c_i$ . Serre showed, using the spectral sequence associated with a presentation of the ring  $A$  as a quotient ring of a regular local ring, that the following inequality holds for the Poincaré series  $F_A(t)$  of the ring  $A$ :

$$F_A(t) \leq \frac{(1+t)^m}{1 - \sum c_i t^{i+1}} \tag{1}$$

(the notation is from (3)).

In this note we indicate a necessary and sufficient condition, formulated in terms of the homologies of the Koszul complex, under which equality holds in the preceding relation. In doing so, one has to take into account not only the additive and multiplicative structure of the homologies of the Koszul complex, but also their structure as an algebra with certain multiary operations (in general, partial

ones). These operations, as a special case of a certain more general construction, were described by Massey (4).

Relation (1) is equivalent (in the case of equality) to the following recurrence relation between the Betti numbers of the ring  $A$ :

$$b_n = c_1 b_{n-2} + \dots + c_m b_{n-m-1} \quad (n \geq m + 1). \quad (2)$$

But the Betti numbers are equal to the ranks of the corresponding free  $A$ -modules in a minimal projective resolution of the  $A$ -module  $k = A/\mathfrak{m}$ . Therefore relation (2) will be proved if it is established that, with a suitable choice of differentials, the  $n$ -th term  $P_n$  of the minimal resolution can be represented in the form

$$P_n = P_{n-2}^{c_1} + \dots + P_{n-m-1}^{c_m}.$$

**Theorem.** The relation (2) holds if and only if the homology algebra of the Koszul complex has zero multiplication and all Massey operations corresponding to simplices are equal to zero.

We shall use the following notation. Elements of the Koszul complex will always be regarded as homogeneous and denoted by  $\lambda$ . Elements of the minimal resolution will be denoted by  $\pi$ . We choose a certain basis in the homology of the Koszul complex, consisting of homogeneous elements; fixed representatives of the classes of this basis will be denoted by  $z$ . To the elements  $z$  there will correspond certain  $A$ -generators in the minimal resolution, which will be denoted by  $u$ . Superscripts on  $\lambda, \pi, z$  will indicate dimension; the superscript on  $u$  will indicate the dimension of the corresponding element  $z$ , which is one less than the dimension of  $u$ .

We proceed to the construction of the minimal resolution. Let  $P_0 = A$ ,  $\varepsilon : P_0 \rightarrow k$  be the augmentation homomorphism. Put  $P_1 = A^m = \Lambda^1 A^m$ ; let  $e_1, \dots, e_m$  be a basis of the free  $A$ -module  $P_1$ ; set  $d_1 e_i = x_i$ ,  $1 \leq i \leq m$ . Let us find the kernel of the homomorphism  $d_1$ ; this is nothing other than the collection of one-dimensional cycles of the Koszul complex. Let  $\{z\}$  be a system of generators of the one-dimensional homology group of the Koszul complex (more precisely, representatives of the corresponding homology classes); then

$$\text{Ker } d_1 = \sum Az + d_2 \Lambda^2 A,$$

and therefore the term  $P_2$  is represented in the form  $P_2 = P_0^{c_1} + \Lambda^2 A^m$ , where  $P_0^{c_1}$  has as basis the elements  $u$  corresponding to the elements  $z$ , and  $d_2 u = z$ .

Let us first make several steps of the construction for the lower dimensions, and then the construction will be carried out by induction. We find cycles in  $P_2$ . These will be, first, linear combinations of cycles of the form

$$x_i u - e_i z = d e_i u - e_i du, \quad z = du,$$

and, second, cycles from  $\Lambda^2 A^m$ . For the construction of the next term of the minimal resolution it is essential to know that cycles of the first kind can generate only such cycles of the second kind as are homologous to zero in the Koszul complex. A linear combination of cycles of the first kind has the form  $d\pi \cdot u - \pi du$ ,  $\pi \in \Lambda^1 A^m = P_1$ ; it is a cycle of the second kind if  $d\pi = 0$ ; but  $\pi du$ , as the product of two cycles of the Koszul complex, is homologous to zero. Thus the minimal basis of cycles in  $P_2$  is composed of three parts: cycles of the form  $d\pi \cdot u - \pi du$ ; cycles of the Koszul complex not homologous to zero; and, finally,  $d_3 \Lambda^3 A^m$ . Thus  $P_3$  can be represented in the form

$$P_3 = P_1^{c_1} + P_0^{c_2} + \Lambda^3 A^m,$$

where  $P_1^{c_1} = \{\pi^1 u^1\}$ ,  $P_0^{c_2} = \{u^2\}$ . We note that if, for some two homogeneous cycles of the Koszul complex, their product is not homologous to zero, then this means that cycles of the first kind in  $P_2$  generate cycles from  $\Lambda^2 A^m$  not homologous to zero, i.e. that the basis of cycles in  $P_2$  chosen above can be shortened, and this means that the three-dimensional Betti number  $b_3$  of the ring  $A$  is smaller than the quantity dictated by relation (2) in the case of equality. We proceed to the construction of the term  $P_4$ , for which we find cycles in  $P_3$ . Here there are cycles of four sorts:

- 1)  $(d_2 \pi^2)u^1 + y(\pi^2, u^1)$ , where  $y(\pi^2, u^1)$  is some element of  $\Lambda^3 A^m$  satisfying the condition

$$dy(\pi^2, u^1) = d_2 \pi^2 \cdot d_2 u^1;$$

- 2)  $d_1 \pi^1 \cdot u^2 - \pi^1 \cdot d_2 u^2$ ;
- 3) cycles from  $\Lambda^3 A^m$  not homologous to zero;
- 4)  $d_4(\Lambda^4 A^m)$ .

It is again shown that, in view of the zero multiplication in the homologies of the Koszul complex, cycles of the first two sorts generate only elements from  $\Lambda^3 A^m$  homologous to zero; therefore there are four sorts of basic elements in the module of cycles, and accordingly  $P_4$  can be represented in the form

$$P_4 = P_2^{c_1} + P_1^{c_2} + P_0^{c_3} + \Lambda^4 A^m,$$

where

$$P_2^{c_1} = \{\pi^2 u^1\}, \quad P_1^{c_2} = \{\pi^1 u^2\}, \quad P_0^{c_3} = \{u^3\};$$

$$d_4(\pi^2 u^1) = (d_2 \pi^2)u^1 + y(\pi^2, u^1); \quad d_4(\pi^1 u^2) = (d_1 \pi^1)u^2 - (\pi^1)du^2; \quad d_4 u^3 = z^3.$$

Multilinear operations in the homology of the Koszul complex begin to play a role only in the construction of the term  $P_5$  (i.e., in computing the cycles in  $P_4$ ). In  $P_4$  there are cycles of five kinds:

- 1)  $(d_3\pi^3)u^1 - y(\pi^3, u^1)$ , where  $dy(\pi^3, u^1) = y(d\pi^3, u^1)$ ;
- 2)  $(d_2\pi^2)u^2 + y(\pi^2, u^2)$ , where  $dy(\pi^2, u^2) = y(d\pi^2, u^2)$ ;
- 3)  $(d_1\pi^1)u^3 - \pi^1 du^3$ , where  $\pi^1 du^3 = y(\pi^1, u^3)$ ;
- 4) cycles from  $\Lambda^4 A^m$  that are not homologous to zero;
- 5)  $d(\Lambda^5 A^m)$ .

To find a basis of cycles one must again verify that cycles of the first three kinds generate only cycles from  $\Lambda^4 A^m$  that are homologous to zero. For cycles of the second and third kinds this follows from the zero multiplication in  $H_*(K)$ . To prove this for cycles of the first kind, one must prove that the element  $y(d_4\pi^4, u^1)$  (which is, obviously, a cycle) is homologous to zero in  $K$ . For this it is necessary to consider four cases:

- 1)  $\pi^4 = \pi^2 u_1^1$ ; 2)  $\pi^4 = \pi^1 u^2$ ; 3)  $\pi^4 = u^3$ ; 4)  $\pi^4 = \lambda^4$ .

In cases 2)–4) the fact that  $y(d_4\pi^4, u^1)$  is homologous to zero again follows from the zero multiplication in  $H_*(K)$ . Consider the first case. It, in turn, splits into two cases:

- 1)  $\pi^2 = u_2^1$ ; 2)  $\pi^2 = \lambda^2$ . To case 2) the preceding remark is applicable.

Thus, let us consider the case  $\pi^4 = u_2^1 u_1^1$ . We have

$$\begin{aligned} y[d_4(u_2^1 u_1^1), u^1] &= y[(d_2 u_2^1) u_1^1, u^1] + y[y(u_2^1, u_1^1), u^1] = \\ &= d_2 u_2^1 \cdot y(u_1^1, u^1) + y(u_2^1, u_1^1) d_2 u^1, \end{aligned}$$

where

$$d_2 u_2^1 = z_2^1, \quad dy(u_1^1, u^1) = z_1^1 z^1, \quad dy(u_2^1, u_1^1) = z_2^1 z_1^1, \quad d_2 u^1 = z^1,$$

i.e.

$$y[d_4(u_2^1, u_1^1), u^1] = z_2^1 y(u_1^1, u^1) + y(u_2^1, u_1^1) z^1 = \gamma(z_2^1, z_1^1, z^1),$$

where  $\gamma(z_2^1, z_1^1, z^1)$  is the ternary Massey operation. The vanishing of this operation to zero is thus equivalent to the fact that the term  $P_5$  has the form

$$P_5 = P_3^{c_1} + P_2^{c_2} + P_1^{c_3} + P_0^{c_4} + \Lambda^5 A^m.$$

We now pass to the inductive step. The projective resolution is regarded as a differential graded module over the Koszul complex (which is a differential graded algebra). Suppose that the  $k$ -th term of the resolution has the form

$$P_k = P_{k-2}^{c_1} + \dots + P_i^{c_{k-i-1}} + \dots + \Lambda^{kA^m} \quad (0 \leq i \leq k-2; i \geq k-m-1),$$

where  $P_i^{c_{k-i-1}} = \{\pi^i u^{k-i-1}\}$ . Suppose that for  $l \leq k$  a function  $y(\pi^j, u^{l-j-1}) \in \Lambda^{lA^m}$  is defined, linear in the first argument (i.e., linear with respect to multipliers from  $\Lambda A^m$ ) and satisfying the following two conditions:

- 1)  $y(1, u) = du$ ;
- 2)  $d[y(\pi^j, u^{l-j-1})] = y(d_j \pi^j, u^{l-j-1})$ .

Define the differential in  $P_k$  by the formula

$$d_k[\pi^i u^{k-i-1}] = (d_i \pi^i) u^{k-i-1} + (-1)^i y(\pi^i, u^{k-i-1}).$$

Find a basis of cycles in  $P_k$ . It is immediately seen that every cycle is a linear combination of cycles of the form

$$(d\pi^{i+1})u^{k-i+1} + (-1)^{i+1}x,$$

where  $x$  must satisfy the condition  $dx = y(d\pi^{i+1}, u^{k-i-1})$ . Thus, the actual existence of all cycles of the indicated form is ensured by the condition: the element  $y(d\pi^{i+1}, u^{k-i-1})$  is homologous to zero in  $\Lambda^k A^m$ . This same condition ensures that the cycles of the corresponding form in  $P_{k-1}$  form a minimal basis. If this condition is satisfied, then we define  $y(\pi^{i+1}, u^{k-i-1})$  by

$$dy(\pi^{i+1}, u^{k-i-1}) = y(d\pi^{i+1}, u^{k-i-1}).$$

If, for the function  $y(d\pi^{i+2}, u^{k-i-1})$  obtained in this way, it is homologous to zero, then the cycles of the above-indicated form form a basis of cycles in  $P_k$ , and we obtain the term  $P_{k+1}$  again in the form we need.

An explicit expression for the function  $y$  is obtained by a simple induction. Let us first note that, in view of linearity, it is enough to define  $y$  in the case when  $\pi$  has the form of the word  $u_1 \dots u_{n-1}$ . If, up to the corresponding place, the function  $y$  has already been defined, then put

$$y[d(u_1 \dots u_{n-1}), u_n] = \gamma(z_1, \dots, z_n),$$

where  $du_i = z_i$ . By induction one verifies that

$$\begin{aligned} \gamma(z_1, \dots, z_n) &= z_1 y(u_2 \dots u_{n-1}, u_n) + \\ &+ \sum_{i=2}^{n-2} (-1)^{\sum_{j=1}^{i-1} (\deg z_j + 1)} y(u_1 \dots u_{i-1}, u_i) y(u_{i+1} \dots u_{n-1}, u_n) + \\ &+ (-1)^{\sum_{j=1}^{n-2} (\deg z_j + 1)} y(u_1 \dots u_{n-2}, u_{n-1}) z_n. \end{aligned}$$

But this is precisely the  $n$ -ary Massey operation. If it is equal to zero (i.e. the corresponding element in  $\Lambda A^m$  is homologous to zero), then one can define the function  $y$  for the next step:

$$dy(u_1 \dots u_{n-1}, u_n) = \gamma(z_1, \dots, z_n).$$

If some product or some operation in the homologies of the Koszul complex is not equal to zero, then the “place” in the construction of the resolution at which this nonzero product or operation first “appears” will precisely give a violation of equality in (1).

An example of rings for which all Massey operations vanish is provided by free reduced nilpotent algebras (i.e. algebras of the form  $A_{d,r} = k[X_1, \dots, X_d]/I$ , where  $k$  is a field and  $I$  is the ideal of polynomials of degree  $\geq r$ ). For the algebra  $A_{d,r}$  the homologies of the Koszul complex are easily computed; it turns out that

$$c_i = \binom{i+r-2}{r-1} \binom{d+r-1}{i+r-1}.$$

The vanishing of the Massey operations follows from the fact that in the homology classes one can choose representatives such that their products are not only homologous to zero, but even equal to zero. Therefore the Poincaré function for the algebra  $A_{d,r}$  is equal to

$$F_{A_{d,r}}(t) = \frac{(1+t)^d}{1 - \sum_{i=1}^d \binom{i+r-2}{r-1} \binom{d+r-1}{i+r-1} t^{i+1}}.$$

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