MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE NUMBER OF DIVISORS IN AN ANGLE

(Presented by Academician P. S. Aleksandrov on 23 X 1961)

The estimates for the mean number of divisors of numbers less than x are well known, namely

$$\sum_{n\leq x}\tau(n)=x\ln x+(2E-1)x+O(x^{\rho}),$$

where, with the aid of estimates of $\zeta(s)$ on the line $\operatorname{Re} s=1/2$, one obtains directly the value $1/3$ for ρ, and this value can be improved. In the present note results are given concerning

$$\sum_{\substack{N(\alpha)\leq x\\ \varphi_1\leq\arg\alpha\leq\varphi_2}}\tau(\alpha),$$

where α is a number of an imaginary quadratic field; the results are based on estimates of functions representable by Hecke series in the critical strip. Let us first note that it is not difficult to prove that

$$\sum_{N(\alpha)=n}\tau(\alpha)=o(n^{\varepsilon})$$

for sufficiently large n.

We shall first consider the fields $k(\sqrt{D})$, where $D\leq-2$, $D\equiv 2,3 \pmod 4$. In these fields there are two units $(1,-1)$, and the basis is the system $1,\sqrt{D}$. Consider the function

$$Z(s)=\sum_{\alpha}\frac{1}{N(\alpha)^s}=\frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}'\frac{1}{(m^2-Dn^2)^s},$$

where $s=\sigma+it$, $\sigma>1$, and $\alpha\neq 0$ runs through all integral nonassociated numbers of the field $k(\sqrt{D})$.

Theorem 1. The function $Z(s)$, defined for $\sigma > 1$, is analytically continuable to the whole s-plane, satisfies the equation

$$(\sqrt{-D})^s \pi^{-s} \Gamma(s) Z(s) = (\sqrt{-D})^{1-s} \pi^{s-1} \Gamma(1-s) Z(1-s)$$

and has a pole of the first order at the point $s = 1$ with residue $\pi/2\sqrt{-D}$.

Proof. We obtain, starting from the function

$$\theta\left(\frac{z}{\sqrt{-D}}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-(m^2 - Dn^2)\pi \frac{z}{\sqrt{-D}}} = \frac{1}{z}\,\theta\left(\frac{1}{z\sqrt{-D}}\right),$$

and from the integral

$$\int_0^\infty \left[\theta\left(\frac{z}{\sqrt{-D}}\right) - 1\right] z^{s-1}\, dz,$$

which gives

$$2(\sqrt{-D})^s \pi^{-s} \Gamma(s) Z(s) = \frac{1}{s(s-1)} + \int_1^\infty \left[\theta\left(\frac{z}{\sqrt{-D}}\right) - 1\right] \left(z^{s-1} + z^{-s}\right)\, dz.$$

Theorem 2.

$$\lim_{s\to 1}\left(Z(s) - \frac{\pi}{2(s-1)\sqrt{-D}}\right) = C_1,$$

$$C_1 = \frac{3\pi + 32C_{1,1}}{4\sqrt{-D}} + \frac{D-1}{D}\,\frac{\pi^2}{6} - 16DJ(1),$$

where

$$C_{1,1} = \int_0^{\varphi_0} 2\sin^2\varphi \cos^2\varphi \ln\sin\varphi\, d\varphi + \int_{\varphi_0}^{\pi/2} 2\sin^2\varphi\cos^2\varphi \ln\cos\varphi\, d\varphi,$$

$$\varphi_0 = \operatorname{arc\,tg}\frac{1}{\sqrt{-D}},$$

$$J(1) = \int_1^\infty \int_1^\infty \frac{xy(\{x\}\{y\} - x\{y\} - y\{x\})}{(y^2 - Dx^2)^3}\, dx\, dy$$

($\{x\}$ is the fractional part of x).

The proof is based on Theorem A (^{3}).

Theorem 3. In the strip $-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$, for large t we have

$$Z(s) = O\left(|t|^{1-\sigma+\varepsilon}\right).$$

The proof of the theorem is based on Theorem 1, on the representation of the function $\Gamma(s)$, and on the theorem of Phragmén–Lindelöf (^{5}).

Theorem 4.

$$\sum_{N(\alpha)\leqslant x} \tau(\alpha) = -\frac{\pi^2}{4D} x \ln x + \left(\frac{\pi^2}{4D} + \frac{C_1 \pi}{\sqrt{-D}}\right) x + O\left(x^{2/3+\varepsilon}\right).$$

For the proof we use Theorem 3 and the lemma on partial sums of the Dirichlet series (^{4}).

Now define the function $Z(s,q)$ as follows:

$$Z(s,q) = \sum_{\alpha} \frac{e^{qi \operatorname{arc} \alpha}}{N(\alpha)^s} = \sum_{\alpha} \frac{\lambda^{2q}(\alpha)}{N(\alpha)^s} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{\left(\frac{m+n\sqrt{D}}{m-n\sqrt{D}}\right)^q}{(m^2-Dn^2)^s},$$

where $q \neq 0$ is an integer rational number, $s = \sigma + it$, $\sigma > 1$.

Theorem 5. The function $Z(s,q)$, defined for $\sigma > 1$, admits analytic continuation to the whole s-plane and satisfies the equation

$$\left(\frac{\sqrt{-D}}{\pi}\right)^s \Gamma(s+|q|) Z(s,q) = \left(\frac{\sqrt{-D}}{\pi}\right)^{1-s} \Gamma(1-s+|q|) Z(1-s,q).$$

The theorem is proved, as is Theorem 1, on the basis of the function

$$\theta_q\left(\frac{z}{\sqrt{-D}}\right) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (m+n\sqrt{D})^{2q} e^{-(m^2-Dn^2)\pi z/\sqrt{-D}} = z^{-2q-1} \theta_q\left(\frac{1}{z\sqrt{-D}}\right).$$

As for the function $Z(s)$, for $Z(s,q)$ we have the following two theorems.

Theorem 6. In the strip $-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$, for large t we have

$$Z(s,q) = O\left[(|t|+|q|)^{1-\sigma+\varepsilon}\right].$$

Theorem 7.

$$\sum_{N(\alpha)\leqslant x} \tau(\alpha)\lambda^{2q}(\alpha) = O\left[(x^{1/3}+|q|)^{2+\varepsilon}\right].$$

Theorem 8.

$$\sum_{\substack{N(\alpha)\leqslant x\\ \varphi_1\leqslant \operatorname{arc}\alpha\leqslant\varphi_2}} \tau(\alpha) = \frac{1}{\pi}(\varphi_2-\varphi_1)\left[-\frac{\pi^2}{4D}x\ln x + \left(\frac{\pi^2}{4D}+\frac{C_1\pi}{\sqrt{-D}}\right)x\right] + O\left(x^{2/3+\varepsilon}\right).$$

The theorem is proved in the same way as Theorem 8 from (^{1}), on the basis of the theorem following from Theorem 7 of (^{1}), in which we assume that $\overline{f(\varphi)}$ and $f(\varphi)$ have period π and the coefficients of their Fourier series satisfy the relations

$$\bar{a}_q \ll \frac{1}{\Delta^3|q|^4}, \qquad a_q \ll \frac{1}{\Delta^3|q|^4}.$$

Now consider the fields $k(\sqrt{D})$, where $D\equiv 1 \pmod 4$, $D<-3$. These fields differ from the preceding ones in that their basis consists of the numbers 1, $\dfrac{1+\sqrt{D}}{2}$. Therefore our functions $Z(s)$ and $Z(s,q)$ have the form:

$$Z(s) = \sum_\alpha \frac{1}{N(\alpha)^s} = \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}' \frac{4^s}{\left[(2m+n)^2-Dn^2\right]^s},$$

$$Z(s,q) = \sum_\alpha \frac{e^{2qi\arg\alpha}}{N(\alpha)^s} = \sum_\alpha \frac{\lambda^{2q}(\alpha)}{N(\alpha)^s} = \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{4^s\left(\dfrac{2m+n+n\sqrt{D}}{2m+n-n\sqrt{D}}\right)^q}{\left[(2m+n)^2-Dn^2\right]^s}.$$

We shall obtain theorems analogous to the preceding ones; here we state the following:

Theorem 8a.

$$\sum_{\substack{N(\alpha)\leq x\\ \varphi_1\leq\arg\alpha\leq\varphi_2}} \tau(\alpha) = \frac{1}{\pi}(\varphi_2-\varphi_1)\left[-\frac{\pi^2}{D}x\ln x + \left(\frac{\pi^2}{D}+\frac{2C_2\pi}{\sqrt{-D}}\right)x\right] + O\left(x^{2/3+\varepsilon}\right),$$

where

$$C_2 = C_2(D) = \lim_{s\to1}\left(Z(s) - \frac{\pi}{(s-1)\sqrt{-D}}\right).$$

The field $k(\sqrt{-3})$, unlike the others, has 6 units. Its basis consists of the numbers $1, \frac{1+\sqrt{-3}}{2}$. Taking this into account and arguing as before, using the results of the case $D \equiv 1 \pmod 4$, we obtain:

Theorem 8b. In the field $k(\sqrt{-3})$,

$$\sum_{\substack{N(\alpha)\le x\\ \varphi_1\le\arg\alpha\le\varphi_2}} \tau(\alpha) = \frac{3}{\pi}(\varphi_2-\varphi_1)\left[\frac{\pi^2}{27}x\ln x + \left(\frac{2\pi C_3}{3\sqrt{3}} - \frac{\pi^2}{27}\right)x\right] + O\left(x^{2/3+\varepsilon}\right),$$

where $C_3 = \frac{1}{3}C_2(-3)$.

For the Gaussian field $k(i)$, which has 4 units, we can argue as for the field $k(\sqrt{-3})$, using the results of the case $D \equiv 2, 3 \pmod 4$. But since the Gaussian field has a single ideal class, we have

$$Z(s) = \sum_{\alpha}\frac{1}{N(\alpha)^s} = \sum_{a}\frac{1}{N(a)^s} = \zeta(s)L(s,\chi),$$

where a runs over all integral ideals of the field,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \qquad L(s,\chi) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^s}.$$

This shows that $Z(s) = \zeta(s)L(s,\chi)$ has a pole of first order at $s = 1$ with residue $\pi/4$, and

$$\lim_{s\to1}\left(\zeta(s)L(s,\chi) - \frac{\pi}{4(s-1)}\right) = C_4 = \frac{\pi E}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n\ln(2n-1)}{2n-1},$$

where E is Euler' s constant.

With the aid of this result and arguments analogous to the preceding ones, we obtain:

Theorem 8c. In the Gaussian field $k(i)$ we have

$$\sum_{\substack{N(\alpha)\le x\\ \varphi_1\le\arg\alpha\le\varphi_2}} \tau(\alpha) = \frac{2}{\pi}(\varphi_2-\varphi_1)\left[\frac{\pi^2}{16}x\ln x + \left(\frac{C_4\pi}{2} - \frac{\pi^2}{16}\right)x\right] + O\left(x^{2/3+\varepsilon}\right).$$

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Note: Figure translations are in progress. See original paper for figures.