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Abstract

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Difference Methods for Solving the Cauchy Problem for the Laplace Equation

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1. It is known that the Cauchy problem for elliptic equations and systems is ill-posed (small perturbations of the initial conditions may cause large perturbations of the solution). However, this problem, like certain other problems that are also ill-posed in the usual sense, arises naturally in various applied questions. In justifying the applied significance of such problems and in constructing approximate methods, the concept of well-posedness in the sense of A. N. Tikhonov is of fundamental importance ([1,2]). A problem is called **well-posed in the sense of A. N. Tikhonov** if uniqueness of the solution holds and the solution depends continuously on the data of the problem under an appropriate a priori restriction on the class of solutions under consideration.

In ([2]), the well-posedness in the sense of A. N. Tikhonov of a number of problems, ill-posed in the usual sense, for the Laplace equation was investigated, and some approximate methods were constructed. These methods, because of their analytic character, cannot be used directly for the numerical solution of analogous problems in the case of linear equations with variable coefficients and nonlinear equations. For solving such equations, difference methods are more convenient.

In the present note, using as an example one boundary-value problem modeling the Cauchy problem for the Laplace equation, we consider certain theoretical and practical questions concerning the application of difference schemes to the solution of problems for elliptic equations and systems that are well-posed in the sense of A. N. Tikhonov.

2. Consider the boundary-value problem:

$$u_{xx} + u_{yy} = 0; \tag{1}$$

$$u(0, y) = u(\pi, y) = 0; \tag{2}$$

$$u(x, 0) = \varphi(x), \qquad u_y(x, 0) = \psi(x). \tag{3}$$

Suppose that in the domain $\{0 \le x \le \pi,\ 0 \le y \le Y\}$ there exists a solution of this problem satisfying the condition

$$\int_0^\pi u^2(x,y)\,dx \le M^2 \qquad (0 \le y \le Y). \tag{4}$$

By the Fourier method this solution may be obtained in the form

$$u(x,y) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} a_k e^{ky} \sin kx + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} b_k e^{-ky} \sin kx. \tag{5}$$

To simplify the formulation of the results, in what follows we shall everywhere assume that the second term on the right-hand side of (5), containing exponents decreasing in y and k, is absent, i.e. $b_k = 0$.

Well-posedness in the sense of A. N. Tikhonov of the boundary-value problem (1)–(3) under the a priori assumption (4) follows from the estimate

$$\left[\int_0^\pi u^2(x,y)\,dx\right]^{\frac{1}{2}} \le \left[\int_0^\pi u^2(x,0)\,dx\right]^{\frac{1}{2}\left(1-\frac{y}{Y}\right)} \left[\int_0^\pi u^2(x,Y)\,dx\right]^{\frac{1}{2}\frac{y}{Y}}, \tag{6}$$

which is proved in the same way as was done in (3) for one boundary-value problem close to (1)–(3). Estimate (6), as is easy to see, cannot be improved.

The nature of the dependence of the right-hand side of (6) on the L_2-norm of the function $u(x,0)$ shows that, for studying by the Fourier method the stability of difference schemes with respect to the initial data, norms of L_2 type are inconvenient. Using the a priori estimate (4), one can introduce for the initial data the stronger norm

$$\|\varphi\|_0 = \left[\sum_{k=1}^{\infty} a_k^2 e^{2kY}\right]^{1/2}. \tag{7}$$

Then, putting

$$\|u\| = \sup_{0 \le y \le Y} \left[\int_0^\pi u^2(x,y)\,dx\right]^{1/2},$$

we obtain from (5) an estimate homogeneous with respect to the norms and not containing the a priori assumptions explicitly:

$$\|u\| \le \|\varphi\|_0. \tag{8}$$

To estimate the efficiency of difference schemes and the possible influence of rounding errors, it is of interest to determine to what extent the solution of problem (1)–(3), satisfying the a priori assumption (4), is determined by prescribing approximate values of the function $u(x,0)=\varphi(x)$ on the grid

$$x = x_n = nh, \qquad n = 1, \dots, N-1, \qquad h = \frac{\pi}{N}. \tag{9}$$

We shall be interested only in the values $u(x,y)$ on the grid lines (9) for $0 \le y \le Y$, and we introduce the corresponding mesh norm

$$\|u(y)\|_h = \left[h \sum_{n=1}^{N-1} u^2(x_n, y) \right]^{1/2}. \tag{10}$$

Theorem 1. Let $u(x,y)$ be a solution of problem (1)–(3) with the a priori condition (4), satisfying the additional requirement $b_k = 0,\ k = 1, 2, \dots$. Construct the function

$$\bar{u}(x,y) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{N-1} \bar{a}_k e^{ky} \sin kx,$$

where the coefficients $\bar{a}_k$ are found from the conditions

$$\bar{u}(x_n, 0) = u(x_n, 0) + \delta_n, \qquad n = 1, 2, \dots, N-1.$$

Then

$$\|\bar{u}(y) - u(y)\|_h \le \sqrt{2M}\, \frac{e^{-(N+1)\eta}}{\sqrt{1 - e^{-4N\eta}}} + \delta e^{(N-1)y}, \tag{11}$$

where

$$\eta = Y - y, \qquad \delta = \left[h \sum_{n=1}^{N-1} \delta_n^2 \right]^{1/2}.$$

3. We shall consider here only difference schemes arising in the approximation of the corresponding boundary-value problem for the Cauchy–Riemann system of equations. We assume the grid to be rectangular and uniform, with steps $\Delta x = h$, $\Delta y = \tau$; $\tau = \tau(h)$, $\tau(h) \to 0$ as $h \to 0$. For such schemes we define the norm of the initial conditions by analogy with (7):

$$\|\varphi\|_{0h} = \left[\sum_{k=1}^{N-1} \hat{a}_k^2 e^{2kY} \right]^{1/2},$$

where $\hat{a}_k$ are the coefficients in the expansion of the grid function $\varphi_n = \varphi(nh)$ in the system of functions

$$\left\{ \sqrt{\frac{2}{\pi}} \sin knh \right\}, \quad k = 1, 2, \dots, N-1.$$

We shall call a two-layer difference scheme stable with respect to the initial conditions if, for $0 \le y \le Y$ and $0 < h \le h_0$, the equality

$$\|u(y)\|_h \le K \|\varphi\|_{0h}, \tag{12}$$

holds, where K is a constant independent of h. For difference schemes admitting solutions of the form $u(mh, n\tau) = u_m^n = s^n(k;h) \sin kmh$, from (12) one directly obtains the necessary condition for stability with respect to the initial data:

$$|s^n(k;h)| \le K e^{kY}, \qquad 0 \le n\tau \le Y. \tag{13}$$

The inequality

$$|s^n(k;h)| \le K e^{ky}, \qquad 0 \le n\tau \le y, \tag{14}$$

will be called the strengthened condition of stability with respect to the initial data.

Theorem 2. *Suppose that a two-layer difference scheme, for any integer N, has a system of solutions $u_m^n = s^n(k;h) \sin kmh$, $k = 1, 2, \dots, N-1$, and that the strengthened stability condition (14) is satisfied. Suppose further that the following consistency condition is satisfied:*

$$\text{as } h \to 0 \qquad \frac{1}{\tau} \ln s(k;h) \to k$$

uniformly in every finite interval $0 \le k \le k_0$. Then the approximate solution u_h converges in the norm (10) to the exact solution.

4. We give the results of the investigation of stability with respect to the initial data for several difference schemes.

A. Explicit schemes.

Scheme 1.

$$\frac{u_m^{n+1}-u_m^n}{\tau}=-\frac{v_{m+1}^n-v_{m-1}^n}{2h},\qquad \frac{v_m^{n+1}-v_m^n}{\tau}=\frac{u_{m+1}^n-u_{m-1}^n}{2h}.$$

For any constant $c=\tau/h$, the strengthened stability condition (14) is satisfied. The approximation error is $e=O(\tau)+O(h^2)$.

Scheme 2.

$$\frac{u_m^{n+1}-u_m^n}{\tau}=-\frac{v_{m+1}^n-v_{m-1}^n}{2h}-\frac{\tau}{2}\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2},$$

$$\frac{v_m^{n+1}-v_m^n}{\tau}=\frac{u_{m+1}^n-u_{m-1}^n}{2h}-\frac{\tau}{2}\frac{v_{m+1}^n-2v_m^n+v_{m-1}^n}{h^2}.$$

Condition (14) is satisfied for any constant $c=\tau/h$. The approximation error is $e=O(\tau^2)+O(h^2)$.

Scheme 3 (leapfrog).

$$\frac{\bar{u}_m^{n+1}-u_m^n}{\tau}=-\frac{v_{m+1}^n-v_{m-1}^n}{2h},\qquad \frac{\bar{v}_m^{n+1}-v_m^n}{\tau}=\frac{u_{m+1}^n-u_{m-1}^n}{2h},$$

$$\frac{u_m^{n+1}-u_m^n}{\tau}=-\frac{1}{2}\left[\frac{v_{m+1}^n-v_{m-1}^n}{2h}+\frac{\bar{v}_{m+1}^{n+1}-\bar{v}_{m-1}^{n+1}}{2h}\right],$$

$$\frac{v_m^{n+1}-v_m^n}{\tau}=\frac{1}{2}\left[\frac{u_{m+1}^n-u_{m-1}^n}{2h}+\frac{\bar{u}_{m+1}^{n+1}-\bar{u}_{m-1}^{n+1}}{2h}\right].$$

The approximation error is $e=O(\tau^2)+O(h^2)$. Condition (14) is satisfied for any constant $c=\tau/h$.

Scheme 4 (three-layer).

$$\frac{u_m^{n+1}-u_m^{n-1}}{2\tau}=-\frac{v_{m+1}^n-v_{m-1}^n}{2h},\qquad \frac{v_m^{n+1}-v_m^{n-1}}{2\tau}=\frac{u_{m+1}^n-u_{m-1}^n}{2h}.$$

Condition (14) is satisfied for any constant $c=\tau/h$.

Scheme 5. A many-point scheme of high accuracy (with respect to h):

$$\frac{u_m^{n+1}-u_m^n}{\tau}=-\frac{1}{h}\sum_{j=-i}^{i}a_j^{(i)}v_{m+j}^n,\qquad \frac{v_m^{n+1}-v_m^n}{\tau}=\frac{1}{h}\sum_{j=-i}^{i}a_j^{(i)}u_{m+j}^n.$$

The approximation error is $e = O(\tau) + O(h^{2i})$ $(i = 1, 2, ...)$. (The coefficients $a_j^{(i)}$ for $i = 1, 2, 3, 4$ are given in ([4]), § 129; a method for computing these coefficients in the general case is indicated there as well.) Condition (14) is satisfied for every i and for any constant $c = \tau/h^*$.

B. Implicit schemes.

Scheme 6.

$$\frac{(u_{m+1}^{n+1} + u_m^{n+1}) - (u_{m+1}^n + u_m^n)}{2\tau} = -\frac{(v_{m+1}^{n+1} + v_{m+1}^n) - (v_m^{n+1} + v_m^n)}{2h},$$

$$\frac{(v_{m+1}^{n+1} + v_m^{n+1}) - (v_{m+1}^n + v_m^n)}{2\tau} = \frac{(u_{m+1}^{n+1} + u_{m+1}^n) - (u_m^{n+1} + u_m^n)}{2h}.$$

For any constant $c = \tau/h$, condition (14) is not satisfied. Let $\tau = O(h^\beta)$; for $\beta < 2$ condition (14) is not satisfied.

Scheme 7.

$$\frac{u_m^{n+1} - u_m^n}{2\tau} = -\frac{1}{2}\left[\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} + \frac{v_{m+1}^n - v_{m-1}^n}{2h}\right],$$

$$\frac{v_m^{n+1} - v_m^n}{2\tau} = \frac{1}{2}\left[\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} + \frac{u_{m+1}^n - u_{m-1}^n}{2h}\right].$$

Put $\tau = O(h^\beta)$. For $\beta < 4/3$ condition (14) is not satisfied; for $\beta > 4/3$ it is satisfied.

5. Using condition (14), for all the schemes considered above, in the presence of enhanced stability one can obtain estimates of the rounding error that depend exponentially on $1/h$. Comparing these estimates with estimates of the approximation error (these errors are proportional to powers of h), one can judge the magnitude of the step h at which the rounding error is comparable with the approximation error. This step, for a given scheme and a prescribed number of digits retained in the computation, determines the greatest attainable accuracy. Such an analysis makes it possible to substantiate the advantages of schemes of high order of accuracy with respect to h (see Scheme 5), which were discovered empirically in the solution of gas-dynamics problems ([5]).

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CITED LITERATURE

[1] A. N. Tikhonov, DAN, **39**, No. 5 (1943).
[2] M. M. Lavrent' ev, Doctoral dissertation, Novosibirsk, 1961.
[3] M. M. Lavrent' ev, Izv. AN SSSR, ser. matem., **20**, No. 6 (1956).
[4] Sh. E. Mikeladze, *Numerical Methods of Mathematical Analysis*, Moscow, 1953.
[5] M. D. Van Dyke, Aero-Space Sci., **25**, No. 9, 485 (1958); Russian translation in *Mekhanika*, No. 5 (51) (1958).

* The stability of this scheme for arbitrary i was proved by V. P. Kudryavtsev (diploma thesis, Moscow State University, 1962).

Note: Figure translations are in progress. See original paper for figures.