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## Abstract

## Full Text

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# EMBEDDING THEOREMS FOR FUNCTIONS FROM THE SPACES $L_p^r$

(Presented by Academician S. L. Sobolev on 27 XI 1961)

1. In recent years, intensive studies have been carried out on the construction of various scales of functional spaces that take into account the differential properties of functions in different metrics. In particular, much attention has been given to the extension and generalization of the well-known  $W_p^{(l)}$ -classification of functional spaces due to S. L. Sobolev <sup>(1)</sup>. At present, the theories of  $H_p^{(r)}$ -spaces (S. M. Nikol'skii <sup>(2)</sup>),  $W_p^{(r)}$ - and  $B_p^{(r)}$ -spaces (N. Aronszajn <sup>(6)</sup>, L. N. Slobodetskii <sup>(3)</sup>, E. Gagliardo, O. V. Besov <sup>(4)</sup>, S. V. Uspenskii <sup>(5)</sup>, P. I. Lizorkin, V. A. Solonnikov, J. L. Lions, and others; see the bibliography in <sup>(4)</sup>) have been created and worked out in detail. The spaces  $W_p^{(r)}$  and  $B_p^{(r)}$  coincide with one another for noninteger  $r$ , and for  $p = 2$  also for integer  $r = l$ , and in this latter case they constitute the natural extension of Sobolev's  $W_2^{(l)}$ -classification to noninteger indices of differentiation. However, for  $p \neq 2$  and noninteger  $r$ ,  $W_p^{(r)}$  do not coincide with the spaces of functions whose Liouville derivatives of order  $r$  are summable to the power  $p$ . Below we propose a classification of functional spaces which is a direct continuation of Sobolev's  $W_p^{(l)}$ -classification for all  $p > 1$ .

The general course of our constructions is as follows. First we introduce the definition of partial derivatives of arbitrary order of generalized functions over a certain space of basic functions. Keeping the usual notation for these derivatives, we then consider classes of functions  $L_{p_1, \dots, p_n}^{(r_1, \dots, r_n)}$  for which the unmixed derivatives of orders  $r_1, \dots, r_n$  with respect to the variables  $x_1, \dots, x_n$  are summable to the powers  $p_1, \dots, p_n$ , respectively. If, in addition, the function itself is summable to the power  $p_0$ , then we obtain the class  $L_{p_0, p_1, \dots, p_n}^{(r_1, \dots, r_n)}$ . Restricting ourselves to the case  $p_0 = p_1 = \dots = p_n = p$ ,  $r_1 = \dots = r_n = r$ , we prove that the class  $L_{p, p, \dots, p}^{(r, \dots, r)}$  coincides, for  $1 < p < \infty$ , with the class  $L_p^r$  of functions representable by a Bessel potential\*. In this case it turns out that the mixed derivatives of order  $r$  of functions  $f \in L_p^r$  exist and their norms in the metric  $L_p$  are estimated in terms of the norm of  $f$  in  $L_p^r$ . Thus, for  $1 < p < \infty$  and integer  $r = l$ , the classes  $L_p^l$  coincide with the spaces  $W_p^{(l)}$  of S. L. Sobolev. By supplementing the classes  $L_p^r$  for  $p = 1$  and  $p = \infty$  with the corresponding Bessel potentials, we prove embedding theorems for them analogous to the theorems of S. M. Nikol'skii for the  $H_p^{(r)}$ -classes, which are a generalization of the known embedding

theorems of S. L. Sobolev and V. I. Kondrashov. Both from the point of view of the original definition and by virtue of the indicated embedding theorems, the classes  $L_p^r$  represent a natural extension of the  $W_p^{(l)}$ -classification to noninteger indices of differentiation.

2. In the Euclidean space  $\widetilde{E}_n(\lambda_1, \dots, \lambda_n)$ , consider the space  $\Psi$  of basic functions ((<sup>10</sup>), p. 101), consisting of all infinitely differenti-

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\* The study of the classes  $L_p^r$  was begun by Aronszajn and Calderón; see papers (6-9).

differentiable functions  $\psi(\lambda)$ , tending to zero together with all their derivatives at infinity and on any coordinate hyperplane (of  $n - 1$  dimensions), such that the quantities

$$\|\psi\|_k = \sup_{l_1 + \dots + l_n \leq k} M_k(\lambda) |D^l \psi(\lambda)|, \quad k = 0, 1, \dots,$$

are finite, where

$$M_k(\lambda) = \max \left\{ (1 + |\lambda|^2)^{k/2}, \frac{1}{\Lambda^k} \right\}, \quad \Lambda = \min \{ \min_i |\lambda_i|, 1 \},$$

and

$$D^l \psi(\lambda) = \frac{\partial^l \psi}{\partial \lambda_1^{l_1} \dots \partial \lambda_n^{l_n}}.$$

Introducing into  $\Psi$  the topology by means of the countable system of norms  $\|\psi\|_k$  turns  $\Psi$  into a complete countably normed space of type  $\mathcal{K}\{M_k\}$  ((<sup>10</sup>), p.115). It is easy to see that if  $\psi(\lambda) \in \Psi$ , then  $(i\lambda)^r \psi(\lambda)$ , where

$$(i\lambda)^r = (i\lambda_1)^{r_1} (i\lambda_2)^{r_2} \dots (i\lambda_n)^{r_n},$$

also belongs to  $\Psi$ . It follows that functionals of the form  $(i\lambda)^r$  are multipliers in the space  $\Psi'$  of generalized functions over  $\Psi$ . Next consider the space  $\Phi$  dual to  $\Psi$ , consisting of functions  $\varphi(x)$  (defined in  $E_n(x_1, \dots, x_n)$ ) that are Fourier transforms of functions  $\psi$  from  $\Psi$ . As is known, the space  $\Phi$ , being dual to the basic space, is also a basic space ((<sup>10</sup>), p.158), and the topology in  $\Phi$  can be introduced starting from the linear isomorphism between  $\Phi$  and  $\Psi$ . On the basis of the well-known theorem on convolutions ((<sup>10</sup>), p.179) we find that the functional  $K^r = \widehat{(i\lambda)^r}$  (the symbol  $\widehat{\phantom{x}}$  denotes the inverse Fourier transform) is a continuous convolution operator in the space  $\Phi'$  of generalized functions over  $\Phi$ . In other words, in the space  $\Phi'$  the operation of convolution with the generalized function  $K^r$  is defined:  $f \rightarrow K^r * f$ . Consider, in particular,  $K^\alpha * K^\beta$ . By the same theorem on convolutions we can write

$$K^\alpha * K^\beta = \widetilde{K^\alpha} \cdot \widetilde{K^\beta} = (i\lambda)^\alpha (i\lambda)^\beta = (i\lambda)^{\alpha+\beta}.$$

It follows that

$$K^\alpha * K^\beta = K^{\alpha+\beta}, \quad K^0 = \widehat{1} = \delta(x), \quad K^r * K^{-r} = K^0 = \delta(x),$$

and, consequently, the operators  $K^r$  form, with respect to the vector parameter  $r = (r_1, \dots, r_n)$ , an additive group of transformations of  $\Phi'$ .

Calculations show that for summable functions  $f$ , decreasing sufficiently rapidly at infinity, the convolution  $K^r * f$  for  $r_i \geq 0$  can be represented in the form

$$K^r * f = \frac{1}{\prod_1^n \Gamma([r_i] + 1 - r_i)} \frac{\partial^{[r_1]+\dots+[r_n]+n}}{\partial x_1^{[r_1]+1} \dots \partial x_n^{[r_n]+1}} \times \\ \times \int_{-\infty}^{x_1} dt_1 \dots \int_{-\infty}^{x_n} \frac{f(t) dt_n}{(x_1 - t_1)^{r_1 - [r_1]} \dots (x_n - t_n)^{r_n - [r_n]}}.$$

In the one-dimensional case we thus obtain the Liouville derivative of order  $r_1$ . For a sufficiently smooth function and integer  $r_i$ , this expression reduces to the ordinary derivative

$$\frac{\partial^r f}{\partial x_1^{r_1} \dots \partial x_n^{r_n}}.$$

In what follows we shall call the convolution  $K^r * f$  the generalized derivative of the function  $f$  and denote it by the symbol

$$\frac{\partial^r f}{\partial x_1^{r_1} \dots \partial x_n^{r_n}}.$$

Let us also note that the constructions of this section are close to the constructions of the work <sup>(11)</sup>.

3. Let us now consider the set  $L_p L_{p, \dots, p}^{(r, \dots, r)}$  of those generalized functions  $F$  from  $\Phi'$  which, together with the derivatives  $\partial^r F / \partial x_i^r$  ( $i = 1, \dots, n$ ), are functions summable over  $E_n$  to the power  $p$ . By introducing the norm

$$\|F\|_{L_p L_{p, \dots, p}^{(r, \dots, r)}} = \|F\|_{L_p(E_n)} + \sum_{i=1}^n \left\| \frac{\partial^r F}{\partial x_i^r} \right\|_{L_p(E_n)}$$

this set becomes a Banach space. Using the linearity of the Fourier transform and the well-known theorem of S. G. Mikhlin on multipliers, one can, for  $1 < p < \infty$ , prove that the function  $(1 + |\lambda|^2)^{r/2} \widetilde{F}$  ( $F \in L_p L_{p, \dots, p}^{(r, \dots, r)}$ ) is the Fourier image

of some function  $f(x) \in L_p(E_n)$ . Hence, for  $F(x)$ , the integral representation follows

$$F(x) = \int_{E_n} G_r(x-t)f(t) dt, \quad f(x) \in L_p(E_n), \quad (1)$$

where, for  $r > 0$ ,

$$G_r(x) = \frac{|x|^{\frac{r-n}{n}}}{2^{\frac{n+r-2}{2}} \pi^{n/2} \Gamma(\frac{r}{2})} \mathcal{K}_{\frac{n-r}{2}}(|x|)$$

( $\mathcal{K}_\alpha(t)$  is the Macdonald function).

Let the class of functions arising from the integral representation (1), when  $f(x)$  ranges over  $L_p(E_n)$  ( $1 \leq p \leq \infty$ ), be denoted by  $L_p^r$ . We have thus proved, for  $1 < p < \infty$ ,  $r > 0$ , the embedding\*

$$L_p L_{p,\dots,p}^{(r,\dots,r)} \rightarrow L_p^r$$

(where  $\|f(x)\|_{L_p(E_n)}$  is taken as the norm in  $L_p^r$ ).

The converse fact also holds: a function  $F(x)$ , representable in the form (1) with  $f(x) \in L_p(E_n)$  ( $1 < p < \infty$ ), belongs to  $L_p L_{p,\dots,p}^{(r,\dots,r)}$ . Moreover, such a function  $F(x)$  has all mixed derivatives of order  $r$ , and the norms of these derivatives in  $L_p$  are estimated in terms of the norm of  $F$  in  $L_p L_{p,\dots,p}^{(r,\dots,r)}$ .

4. Let us turn to embedding theorems for the spaces  $L_p^r$ ,  $1 \leq p \leq \infty$ ,  $r \geq 0$  (we agree to regard  $L_p^0 \equiv L_p(E_n)$ ).

First of all, note that the space  $L_\infty^r$ , for  $r > 0$ , consists of continuous functions possessing a certain margin of smoothness characterized by the index  $r$ . For example, when  $0 < r < 1$ , functions in  $L_\infty^r$  satisfy the Hölder condition with exponent  $r$ . In the general case one may assert that functions in  $L_\infty^r$  belong to the space  $H_\infty^r$  of S. M. Nikolskii <sup>(2)</sup> ( $L_\infty^r \rightarrow H_\infty^r$ ).

**Theorem 1.** There is an embedding  $L_p^r(E_n) \rightarrow L_{p'}^{r'}(E_n)$  under the following restrictions on the parameters:

$$\text{a) } r' - \frac{n}{p'} \leq r - \frac{n}{p}, \quad \text{b) } 1 \leq p \leq p' < \infty,$$

where the equality sign in a) should be discarded when either  $(r - r')p = n$  or  $p = 1$ .

A special case of this theorem (for  $r' = 0$  and without considering  $p' = \infty$ ) was proved in <sup>(6)</sup>. The proof is based on Young's theorem on convolutions—

\* Recall that, by definition, a Banach space  $B_1$  is embedded in a Banach space  $B_2$  ( $B_1 \rightarrow B_2$ ), if  $B_1 \subset B_2$  and there exists a constant  $c$ , common to all elements  $f \in B_1 \subset B_2$ , such that

$$\|f\|_{B_2} \leq c\|f\|_{B_1}.$$

and uses estimates of integrals of potential type due to S. L. Sobolev.

Let now  $E_m$  be an  $m$ -dimensional hyperplane obtained by fixing the coordinates  $x_{m+1}, \dots, x_n$ . We shall consider the function  $F(x_1, \dots, x_n) \in L_p^r(E_n)$  as a function only of the variables  $x_1, \dots, x_m$ . The question is whether the function thus obtained,  $F'(x_1, \dots, x_m) = F'(x')$ , can be characterized in terms of the spaces  $L_{p'}^{r'}(E_m)$ . The answer is given by the following embedding theorem.

**Theorem 2.** Let  $F(x) \in L_p^r(E_n)$ ,  $1 \leq p \leq \infty$ . Then the function  $F'(x') \in L_{p'}^{r'}(E_m)$  under the following restrictions on the indices:

$$\text{a) } r' - \frac{m}{p'} \leq r - \frac{n}{p}, \quad \text{b) } p < p' < \infty,$$

where the equality sign in a) is to be excluded when  $p = 1$ .

In the proof of this theorem, use was made in an essential way of an inequality of V. P. Il' in <sup>12</sup>, somewhat modified by us.

We also note that, in applications to boundary-value problems for partial differential equations, statements about derivatives of the function  $F$  "normal" to  $E_m$  also acquire an important role. These statements are obtained on the basis of Theorem 2, taking into account the fact that each differentiation lowers by one the upper index of the space  $L_p^r(E_n)$ ,  $1 < p < \infty$ , to which the function belongs.

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*Note: Figure translations are in progress. See original paper for figures.*

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