ON VARIATIONAL METHODS IN THE NONLINEAR THEORY OF ELASTICITY

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Abstract

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ON VARIATIONAL METHODS IN THE NON-LINEAR THEORY OF ELASTICITY

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In constructing approximate solutions of problems in the theory of elasticity, it is convenient to use variational methods, which make it possible to avoid the often insurmountable difficulties of finding solutions of differential equations describing the behavior of an elastic body under the corresponding boundary conditions.

However, the principle of virtual displacements (Lagrange' s variational equation), the principle of virtual changes of the stress state (Castigliano's principle), and the Ritz method adjoining them rely in an essential way on one or another set of constraints imposed on the geometric and static characteristics inside and on the boundary of the body. It is expedient to free oneself from these conditions, which greatly complicate the calculation and the selection of functions in constructing approximate solutions.

In the linear formulation for a three-dimensional body and in the geometrically nonlinear formulation for thin plates, this problem was solved in the work of Hu Haichang ([1]). For shallow thin shells these questions were partially solved in the works of N. A. Alumyae ([2]) and, in a more general formulation, in the work of L. Ya. Ainola ([3]). The same questions were earlier studied by E. Reissner ([4,5]) and de Veubeke ([6]). Works by K. Z. Galimov ([7,8]) are devoted to the development of general variational methods.

In the present work a principle is formulated from which all the relations of the nonlinear theory of elasticity follow, and the independently varied quantities of displacements, stresses, and strains are free of any constraints inside and on the boundary of the body. In what follows, several special cases of this principle are written down, corresponding to the known satisfaction of one or another relation of the nonlinear theory of elasticity.

1. Consider the functional

$$I = \iiint\limits_V \mathbf{Q}\mathbf{u}\,dV + \iint\limits_{S_p} \mathbf{P}_S\mathbf{u}\,dS + \iint\limits_{S_u} \sigma^{ik} r_k^* n_i (\mathbf{u}_S - \mathbf{u})\,dS -$$

$$-\iiint\limits_V \left\{W - \sigma^{ik}\left[\varepsilon^*_{ik} - \frac{1}{2}\left(r^*_i\partial_k\mathbf{u} + r^*_k\partial_i\mathbf{u} - \partial_i\mathbf{u}\,\partial_k\mathbf{u}\right)\right]\right\} dV. \tag{1}$$

Here V is the volume occupied by the body before deformation;

$$\mathbf{Q} = \mathbf{Q}_*\sqrt{\frac{g_*}{g}}, \qquad \mathbf{P}_S = \mathbf{P}^*_S\sqrt{\frac{a^*_S}{a_S}}; \qquad \sigma^{ik} = \sigma^{ik}_*\sqrt{\frac{g_*}{g}}; \qquad W = W_*\sqrt{\frac{g_*}{g}};$$

$$g = \det(g_{ik}); \qquad g_* = \det(g^*_{ik}); \qquad (g_{ik} = \mathbf{r}_i, \mathbf{r}_k); \qquad g^*_{ik} = (\mathbf{r}^*_i, \mathbf{r}^*_k); \tag{2}$$

$\mathbf{r}_i$ are the coordinate vectors in the undeformed space referred to the parametrization x^i $(i = 1, 2, 3)$; $\mathbf{r}^*_i = \mathbf{r}_i + \partial_i\mathbf{u}$ are the coordinate vectors in the deformed volume V_*, referred to the parametrization x^i, into which, in the process of deformation, the parametrization of the undeformed body passes; $\mathbf{u}$ is the displacement vector; $\mathbf{Q}_*$ is the vector of body forces,

referred to a unit of deformed volume; $\mathbf{P}^*_S$ are external surface loads referred to a unit area on S_* (S_* is the boundary of V_*) and prescribed on the part S^*_p of the surface S_*; $\mathbf{u}_S$ are displacements prescribed on the part S^*_u of the surface S_*; $\sigma^{ik}_* = \sigma^{ki}_*$ are the contravariant components of the stress tensor referred to the deformed state; n_i are the covariant components of the unit vector of the interior normal to the surface S (S is the boundary of V); $W_* = W_*(\varepsilon^*_{ik})$ is the strain-energy density in a unit of deformed volume, which is a function only of the strain tensor; ε^*_{ik} are the covariant components of the strain tensor, $\partial_i(...) = \partial(...)/\partial x^i$.

In what follows we shall assume that the variations of the body forces referred to a unit of initial volume, and the variations of the external surface loads referred to a unit area on S, are equal to zero, i.e. $\delta\mathbf{Q} = \delta\mathbf{P}_S = 0$. This imposes no restrictions on the generality of the results set forth below.

The following assertion holds: among all displacements, stresses, and strains, only those actually occur which impart to the functional I a stationary value.

Relying on the fact that [8]

$$\delta r^*_i = \delta\partial_i\mathbf{u}, \qquad \sigma^{ik}r^*_k\partial_i\delta\mathbf{u} = \sigma^{ik}\delta\frac{1}{2}\left(r^*_i\partial_k\mathbf{u} + r^*_k\partial_i\mathbf{u} - \partial_i\mathbf{u}\,\partial_k\mathbf{u}\right), \qquad \iint_S = \iint_{S_p} + \iint_{S_u}, \tag{3}$$

and using the formula for passing from volume integration to integration over the surface, for the first variation of the functional I we obtain

$$\delta I = \iiint_V \{\nabla_i \sigma^{ik} r_k^* + \mathbf{Q}\} \, \delta\mathbf{u} \, dV + \iint_{S_p} (\mathbf{P}_S + \sigma^{ik} r_k^* n_i) \, \delta\mathbf{u} \, dS +$$

$$+ \iint_{S_u} (\mathbf{u}_S - \mathbf{u}) \, \delta(\sigma^{ik} r_k^* n_i) \, dS + \iiint_V \left\{ \sigma^{ik} - \frac{\partial W}{\partial \varepsilon_{ik}^*} \right\} \delta\varepsilon_{ik}^* \, dV +$$

$$+ \iiint_V \left\{ \varepsilon_{ik}^* - \frac{1}{2} (r_i^* \partial_k \mathbf{u} + r_k^* \partial_i \mathbf{u} - \partial_i \mathbf{u} \, \partial_k \mathbf{u}) \right\} \delta\sigma^{ik} \, dV. \tag{4}$$

Here $\nabla_i(...)$ denotes covariant differentiation with respect to the metric g_{ik}. Let W be the stress potential. Then from the expression for δI it is immediately clear that, when all relations of the nonlinear theory of elasticity are satisfied, we have $\delta I = 0$. On the other hand, by virtue of the independence of the variations of displacements, stresses, and strains inside and on the boundary of the body, we obtain all the relations of the nonlinear theory of elasticity from the condition $\delta I = 0$: the equations of equilibrium

$$\nabla_i \sigma^{ik} r_k^* + \mathbf{Q} = 0; \tag{5}$$

the elastic relations

$$\sigma^{ik} = \frac{\partial W}{\partial \varepsilon_{ik}^*}, \tag{6}$$

the strain-compatibility conditions

$$2\varepsilon_{ik} = r_i^* \partial_k \mathbf{u} + r_k^* \partial_i \mathbf{u} - \partial_i \mathbf{u} \, \partial_k \mathbf{u} = r_i \partial_k \mathbf{u} + r_k \partial_i \mathbf{u} + \partial_i \mathbf{u} \, \partial_k \mathbf{u}; \tag{7}$$

the natural static boundary conditions on S_p

$$\mathbf{P}_S + \sigma^{ik} r_k^* n_i = 0; \tag{8}$$

the natural geometric boundary conditions on S_u

$$\mathbf{u}_S - \mathbf{u} = 0. \tag{9}$$

It is not difficult to show that the functional remains valid also in the case of mixed boundary conditions on some part of the surface S, i.e., when on S_{pu} part of the conditions are prescribed geometrically and part statically.

2. Let the equilibrium equations be satisfied. This can be achieved if, to a particular solution $F^i_{(r)}$ of the equation

$$\nabla_i F^i + Q = 0$$

one adds the general solution of the homogeneous equation

$$\nabla_i F^i = 0.$$

The latter may be taken in the form (7)

$$F^i_{(0)} = c^{imp} c^{knq} \nabla_m \nabla_n \Phi_{pqr_k}. \tag{10}$$

Then the general solution of the equilibrium equations assumes the form

$$\sigma^{ik} = c^{imp} c^{rnq} \left(\delta^k_r + \nabla_r u^k\right) \nabla_m \nabla_n \Phi + F^i_{(r)} r^k * . \tag{11}$$

Here $\mathbf{u} = u^k r_k$.

In this case we obtain

$$I_1 = \iint_{S_p} \left(P_S + \sigma^{ik} r^*_k n_i\right) u\, dS + \iint_{S_u} \sigma^{ik} r^*_k n_i u_S\, dS - \iiint_V \left\{W - \sigma^{ik}\left(\varepsilon^*_{ik} + \frac{1}{2}\partial_i u\, \partial_k u\right)\right\} dV. \tag{12}$$

3. If, along with satisfaction of the equilibrium equations, the elasticity relations are fulfilled, i.e.,

$$\sigma^{ik} = \frac{\partial W}{\partial \varepsilon^*_{ik}}, \tag{13}$$

then, for the linear Hooke law

$$\sigma^{ik} = A^{ikmn} \varepsilon^*_{mn} \tag{14}$$

we obtain

$$I_2 = \iint_{S_p} \left(P_S + \sigma^{ik} r^*_k n_i\right) u\, dS + \iint_{S_u} \sigma^{ik} r^*_k n_i u_S\, dS + \frac{1}{2} \iiint_V A^{ikmn} \varepsilon^*_{ik} \left(\varepsilon^*_{mn} + \partial_i u \partial_k u\right) dV. \tag{15}$$

This functional contains all boundary conditions as natural ones, as well as the continuity relations for deformation, and its first variation has the form

$$\delta I_2 = \iint_{S_p} \left(P_S + \sigma^{ik} r_k^* n_i\right) \delta u \, dS + \iint_{S_u} (u_S - u)\, \delta \left(\sigma^{ik} r_k^* n_i\right) dS + \iiint_V \left\{ \varepsilon_{ik}^* - \frac{1}{2}\left(r_i^* \partial_k u + r_k^* \partial_i u - \partial_i u \, \partial_k u\right)\right\} \delta\sigma^{ik} \, dV \tag{16}$$

4. Let only the compatibility conditions for deformations be satisfied, and

$$2\varepsilon_{ik}^* = r_i^* \partial_k u + r_k^* \partial_i u - \partial_i u \, \partial_k u. \tag{17}$$

In this case the functional I takes the form:

$$I_3 = \iiint_V Qu \, dV + \iint_{S_p} P_S u \, dS + \iint_{S_u} \sigma^{ik} r_k^* n_i (u_S - u) \, dS - \iiint_V W \, dV, \tag{18}$$

and the first variation gives

$$\delta I_3 = \iiint_V \left\{\nabla_i \sigma^{ik} r_k^* + \mathbf{Q}\right\} \delta \mathbf{u} \, dV + \iint_{S_p} \left(\mathbf{P}_S + \sigma^{ik} r_k^* n_i\right) \delta \mathbf{u} \, dS +$$

$$+ \iint_{S_u} (\mathbf{u}_S - \mathbf{u})\, \delta \left(\sigma^{ik} r_k^* n_i\right) dS + \iiint_V \left(\sigma^{ik} - \frac{\partial W}{\partial e_{ik}^*}\right) \delta e_{ik}^* \, dV. \tag{19}$$

The possibilities of obtaining other particular cases of the functional are evident.

The functional constructed here, and its particular cases, can substantially facilitate the work of solving concrete problems, since the construction of the functional is considerably simpler than the construction of the corresponding variational equations, just as the calculation of energy in the Ritz method is simpler than the construction of the variational equation from which the condition of minimum energy follows. Moreover, the use of the functional is convenient in that it contains all boundary conditions as natural ones, which facilitates the selection of approximating functions. At the same time, the use of general theorems of the mechanics of deformable media may prove fruitful in obtaining general equations, as has been done, for example, in application to the theory of shells in works ([5],[9]).

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CITED LITERATURE

1. Hsü Hsi-chang, *Scientia Sinica*, **4**, No. 1 (1955).

2. N. A. Alumyae, *Applied Mathematics and Mechanics*, **14**, issues 1 and 2 (1950).

3. L. Ya. Ainola, *Proceedings of the Tallinn Polytechnic Institute*, Series A, No. 104 (1957).

4. E. Reissner, *J. Math. Phys.*, **29**, No. 2 (1959).

5. E. Reissner, *J. Math. Phys.*, **23**, 184 (1944).

6. B. de Veubeke, *Bull. Service Techn. et Industr. Aéronaut.*, No. 24, Bruxelles (1951).

7. K. Z. Galimov, *Scientific Notes of Kazan State University*, **115**, book 12, 111 (1955).

8. K. Z. Galimov, *Scientific Notes of Kazan State University*, **109**, book 1, 35 (1949).

9. Kh. M. Mushtari, I. G. Teregulov, *DAN*, **128**, No. 6 (1959).

Note: Figure translations are in progress. See original paper for figures.