MATHEMATICS

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Abstract

Full Text

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ON NONUNIQUENESS FOR PARABOLIC SYSTEMS

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In the work ([1]) I attempted to show that a nonlinear parabolic system of the form

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left[\sum_k a_{ik}\frac{\partial u_k}{\partial x}\right]. \tag{1}$$

need not have a unique solution in the case of discontinuous initial data. S. N. Kruzhkov drew my attention to an error that I had made in my calculations. He also proved that for the system considered in ([1]),

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a(u,v)\frac{\partial u}{\partial x}\right],$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2},$$

the solution is unique also for discontinuous initial data. Nevertheless, the assertion that a nonunique solution is possible for systems of type (1) is correct. We shall now show this for a system of three equations, based on the same idea as in ([1]).

First let us note that the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(\xi)\frac{\partial u}{\partial x}\right] \qquad \left(\xi = \frac{x}{\sqrt{t}}\right)$$

for

$$K(\xi) = \begin{cases} 1, & \text{for } -\infty < \xi < -2, \\ f(\xi), & \text{for } -2 \le \xi \le -1, \\ 1, & \text{for } -1 < \xi < +\infty \end{cases}$$

with the initial condition $u|_{t=0} = \operatorname{sign} x$ is written by the formulas

$$u(\xi) = -1 + 2\int_{-\infty}^{\xi} \frac{z(\eta)}{2K(\eta)}\,d\eta;$$

$$z(\xi) = \begin{cases} A_1 \exp\left[-\dfrac{\xi^2}{4}\right], & \text{for } \xi \le -2, \\ A_2 \exp\left[-\displaystyle\int_{-2}^{\xi} \frac{\eta\,d\eta}{2f(\eta)}\right], & \text{for } -2 < \xi \le -1, \\ A_3 \exp\left[-\dfrac{\xi^2}{4}\right], & \text{for } -1 < \xi. \end{cases}$$

Of the expressions for the constants A_1, A_2, A_3 we shall give only one:

$$A_3 = \left\{\int_{-\infty}^{+\infty} K^{-1}(\xi)\exp\left[-\int_0^{\xi} \frac{\eta\,d\eta}{2K(\eta)}\right] d\xi\right\}^{-1}.$$

The solution of the equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \qquad \left(w|_{t=0} = \operatorname{sign} x\right)$$

is defined as follows:

$$w(\xi) = -1 + 2\int_0^{\xi} \bar{z}(\eta)\,d\eta,$$

where

$$\bar{z}(\xi) = \left[\int_{-\infty}^{+\infty} \exp\left[-\frac{\xi^2}{4}\right] d\xi\right]^{-1} \exp\left[-\frac{\xi^2}{4}\right].$$

It is clear from these formulas that in the plane u, w the curve defined by the parametric equations

$$u = u(\xi), \qquad w = w(\xi),$$

connects the points $(-1,-1)$, $(+1,+1)$ and contains two rectilinear segments. One of these segments corresponds to the interval $-1 < \xi < +\infty$, and on it

$$\frac{du}{dw} : \frac{z(\xi)}{\bar{z}(\xi)} = A_3 \int_{-\infty}^{+\infty} \exp\left[-\frac{\xi^2}{4}\right] d\xi = \text{const} = B.$$

We shall now show that $B < 1$. Since

$$B = \left\{ \int_{-\infty}^{0} \exp\left[-\frac{\xi^2}{4}\right] d\xi + \int_{0}^{+\infty} \exp\left[-\frac{\xi^2}{4}\right] d\xi \right\} \times$$

$$\times \left\{ \int_{-\infty}^{0} \frac{1}{K(\xi)} \exp\left[-\int_0^{\xi} \frac{\eta\, d\eta}{2K(\eta)}\right] d\xi + \int_0^{\infty} \exp\left[-\frac{\xi^2}{4}\right] d\xi \right\}^{-1},$$

it suffices to verify that

$$\int_{-\infty}^{0} \exp\left[-\frac{\xi^2}{4}\right] d\xi < \int_{-\infty}^{0} \frac{1}{K(\xi)} \exp\left[-\int_0^{\xi} \frac{\eta\, d\eta}{2K(\eta)}\right] d\xi. \tag{2}$$

Denote

$$s(\xi) = -\left[2\int_0^{\xi} \frac{\eta\, d\eta}{K(\eta)}\right]^{1/2}.$$

The inverse function satisfies the equation

$$\xi(s) = -\left[2\int_0^{s} K|\xi(\gamma)|\, \gamma\, d\gamma\right]^{1/2}.$$

Since $K(\xi) \le 1$ and $K(\xi) \not\equiv 1$, we have $\xi(s)/s \le 1$ and $\xi(s)/s \not\equiv 1$. The integral appearing on the right-hand side of inequality (3) can be rewritten as

$$\int_{-\infty}^{0} \frac{1}{K(\xi)} \exp\left[-\int_0^{\xi} \frac{\eta\, d\eta}{2K(\eta)}\right] d\xi = \int_{-\infty}^{0} \frac{s}{\xi} \exp\left[-\frac{s^2}{4}\right] ds >$$

$$> \int_{-\infty}^{0} \exp\left[-\frac{s^2}{4}\right] ds = \int_{-\infty}^{0} \exp\left[-\frac{\xi^2}{4}\right] d\xi;$$

thus it is proved that $B < 1$.

It is seen from this that the curve $u = u(\xi), w = w(\xi)$ can have common points with the line $u = w$, distinct from $(+1, +1)$, only for $\xi < -1$.

Defining the function $v(\xi)$ as a solution of the equation

$$\frac{dv}{dt} = \frac{\partial}{\partial x}\left[K(-\xi)\frac{\partial v}{\partial x}\right]$$

with initial data $v|_{t=0} = \operatorname{sign} x$, we find that $v(\xi) = -u(-\xi)$, and thereby establish that the curve $v = v(\xi), w = w(\xi)$ can intersect the line $v = w$ only at points with $\xi > 1$ and at the point $v = w = -1$. Hence it is clear that in the space (u, v, w) the curve $u = u(\xi), v = v(\xi), w = w(\xi)$ intersects the line $u = v = w$ only when $u = v = w = \pm 1$. Let us also note that $u = u(\xi), v = v(\xi), w = w(\xi)$ are strictly monotone functions.

Define along the curve $u = u(\xi), v = v(\xi), w = w(\xi)$ the coefficients

$$a(u, v, w) = K(\xi),$$
$$b(u, v, w) = K(-\xi),$$
$$c(u, v, w) = 1.$$

On the line $u = v = w$ put $a = b = c = 1$. After this we extend a, b, c to the whole space (u, v, w). These extended a, b, c may be regarded as greater than 1/2, not exceeding 1, and sufficiently smooth. The possibility of a smooth extension is ensured by the smoothness of the curve $u = u(\xi), v = v(\xi), w = w(\xi)$. This smoothness can be achieved by choosing a smooth $f(\xi)$.

The system of equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a\frac{\partial u}{\partial x}\right],$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[b\frac{\partial v}{\partial x}\right],$$
$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[c\frac{\partial w}{\partial x}\right]$$

with initial data

$$u|_{t=0} = v|_{t=0} = w|_{t=0} = \operatorname{sign} x$$

has the solutions:

$$1) \quad \begin{cases} u = u(\xi), \\ v = v(\xi), \\ w = w(\xi); \end{cases} \qquad 2) \quad \begin{cases} u = u(\xi), \\ v = u(\xi), \\ w = u(\xi). \end{cases}$$

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CITED LITERATURE

1. S. K. Godunov, DAN, 136, No. 6, 1281 (1961).

Note: Figure translations are in progress. See original paper for figures.