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Abstract

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ON THE SPECTRUM OF ELEMENTARY EXCITATIONS IN THE THEORY OF SUPERCONDUCTIVITY

(Presented by Academician N. N. Bogolyubov, 21 X 1961)

The study of collective excitations in the theory of superconductivity, carried out by N. N. Bogolyubov ([1]), showed that in a neutral Fermi system at zero temperature there exist longitudinal waves possessing the velocity $v_F/\sqrt{3}$ (v_F is the velocity of particles on the Fermi surface). Subsequently N. N. Bogolyubov and P. Anderson obtained small interaction corrections to this value of the velocity of longitudinal waves at zero temperature ([2,3]). In the present article elementary excitations in a Fermi system at a temperature different from zero are considered.

The Fermi system is specified by the four-operator Hamiltonian

$$H = \sum T(f, f')\, a_f^+ a_{f'} + \frac{1}{2} \sum J(f_1, f_2; f_2', f_1')\, a_{f_1}^+ a_{f_2}^+ a_{f_2'} a_{f_1'}, \tag{1}$$

where $T(f, f') = I(f, f') - \lambda\delta(f - f')$ (I is the particle Hamiltonian, λ the chemical potential), while I and J obey the usual conditions of symmetry, Hermiticity, etc. Let us consider the equations for the two-time Green' s functions ([4,5]) $G_{1mnlp}(t, t') = \langle\langle a_m^+ a_n \mid a_l^+ a_p \rangle\rangle$, $G_{2mnlp}(t, t') = \langle\langle a_{ma} n \mid a_l^+ a_p \rangle\rangle$, and carry out, on the right-hand side, a decoupling of higher-order Green' s functions, pairing all operators depending on t two by two. Calculating the commutators and carrying out, according to the indicated principle, all possible decouplings, we obtain

$$
\begin{aligned}
i\frac{dG_{1mnlp}(t,t')}{dt} = {} & \delta(t-t')\{\delta_{nl}F(m,p)-\delta_{mp}F(l,n)\}+ \\
& + \sum\{E(n,f)\langle\langle a_m^+ a_f \mid a_l^+ a_p\rangle\rangle - E(f,m)\langle\langle a_f^+ a_n \mid a_l^+ a_p\rangle\rangle\}+ \\
& + \sum\{J(n,f'';f',f)-J(n,f'';f,f')\}F(m,f)\langle\langle a_{f''}^+ a_{f'} \mid a_l^+ a_p\rangle\rangle- \\
& - \sum\{J(f,f';f'',m)-J(f,f';m,f'')\}F(f,n)\langle\langle a_{f'}^+ a_{f''} \mid a_l^+ a_p\rangle\rangle+ \\
& + \sum\{-S(n,f)\langle\langle a_m^+ a_f^+ \mid a_l^+ a_p\rangle\rangle + S^*(m,f)\langle\langle a_{fa}n \mid a_l^+ a_p\rangle\rangle\}+ \\
& + \sum J(n,f;f'',f')\Phi^*(m,f)\langle\langle a_{f'} a_{f''} \mid a_l^+ a_p\rangle\rangle- \\
& - \sum J(f',f'';m,f)\Phi(n,f)\langle\langle a_{f''}^+ a_{f'}^+ \mid a_l^+ a_p\rangle\rangle;
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
i\frac{dG_{2mnlp}(t,t')}{dt} = {} & \delta(t-t')\{\delta_{nl}\Phi(m,p)-\delta_{ml}\Phi(n,p)\}+ \\
& + \sum\{E(m,f)\langle\langle a_{fa}n \mid a_l^+ a_p\rangle\rangle + E(n,f)\langle\langle a_{ma}f \mid a_l^+ a_p\rangle\rangle\}+ \\
& + \sum J(m,n;f'',f')\langle\langle a_{f'} a_{f''} \mid a_l^+ a_p\rangle\rangle+ \\
& + \sum\{J(n,f'';f',f)-J(n,f'';f,f')\}\Phi(m,f)\langle\langle a_{f''}^+ a_{f'} \mid a_l^+ a_p\rangle\rangle+ \\
& + \sum\{J(m,f'';f',f)-J(m,f'';f,f')\}\Phi(f,n)\langle\langle a_{f''}^+ a_{f'} \mid a_l^+ a_p\rangle\rangle- \\
& - \sum\{S(f,n)\langle\langle a_f^+ a_m \mid a_l^+ a_p\rangle\rangle + S(m,f)\langle\langle a_f^+ a_n \mid a_l^+ a_p\rangle\rangle\}- \\
& - \sum J(f,n;f'',f')F(f,m)\langle\langle a_{f'} a_{f''} \mid a_l^+ a_p\rangle\rangle- \\
& - \sum J(m,f;f'',f')F(f,n)\langle\langle a_{f'} a_{f''} \mid a_l^+ a_p\rangle\rangle,
\end{aligned}
\tag{3}
$$

where

$$
\begin{gathered}
E(f_1,f_2) = T(f_1,f_2) + \sum\{J(f_1,f'';f',f_2)-J(f_1,f'';f_2,f')\}F(f'',f'), \\
S(f_1,f_2) = \sum J(f_1,f_2;f'',f')\Phi(f',f''), \\
F(f_1,f_2) = \langle a_{f_1}^+ a_{f_2}\rangle, \quad \Phi(f_1,f_2) = \langle a_{f_1} a_{f_2}\rangle.
\end{gathered}
$$

Equations (2) and (3), up to inhomogeneous terms, coincide with the variational equations for the correlation functions F and Φ, used by N. N. Bogolyubov to determine the spectrum of elementary excitations of the ground state of a superconducting system. Thus, the variational equations for the correlation functions contain the results of the next approximation in Green' s functions. To solve equations (2) and (3), following [(2)], we shall assume that in the initial Hamiltonian a completely generalized canonical transformation has been carried out, $a_f = \sum(u_{f\nu}\alpha_\nu + v_{f\nu}\alpha_\nu^+)$.

To determine δF and $\delta\Phi$, instead of varying $u_{f\nu}$ and $v_{f\nu}$, we perform an infinitesimal transformation on the operators α_ν: $\alpha_\nu \to \alpha_\nu + \sum \mu(\nu',\nu)\alpha_{\nu'} + \sum \lambda(\nu',\nu)\alpha^+_{\nu'}$. From the condition of canonicity of this transformation it follows that $\lambda(\nu_1,\nu_2)+\lambda(\nu_2,\nu_1)=0$; $\mu^*(\nu_1,\nu_2)+\mu(\nu_2,\nu_1)=0$.

Using the last relations, one can find δF and $\delta\Phi$, and, applying the orthonormality conditions for $u_{f\nu}$ and $v_{f\nu}$, express λ and μ through δF and $\delta\Phi$. Differentiating these relations with respect to time and taking into account the established correspondence and equations (2) and (3), one can obtain a system of equations for determining λ, λ^*, μ, and μ^*:

$$\begin{aligned} i\frac{\partial\lambda(\gamma_1,\gamma_2)}{\partial t} = \frac{1}{1-n_{\gamma_1}-n_{\gamma_2}}\Bigg\{ &\sum [\lambda(\gamma_1,\nu)(1-n_\nu-n_{\gamma_1})\Omega(\gamma_2,\nu) \\ &- \lambda(\gamma_2,\nu)(1-n_\nu-n_{\gamma_2})\Omega(\gamma_1,\nu)]+ \\ &+\sum \lambda(\nu_1,\nu_2)(1-n_{\nu_1}-n_{\nu_2})[Z(\gamma_1,\gamma_2;\nu_1,\nu_2)-Z(\gamma_2,\gamma_1;\nu_1,\nu_2)]+ \\ &+\sum \lambda^*(\nu_1,\nu_2)(1-n_{\nu_1}-n_{\nu_2})[Y(\gamma_1,\gamma_2;\nu_1,\nu_2)-Y(\gamma_2,\gamma_1;\nu_1,\nu_2)]+ \\ &+\sum [\mu(\nu,\gamma_2)(n_\nu-n_{\gamma_2})\beta^*(\nu,\gamma_1)+\mu(\nu,\gamma_1)(n_\nu-n_{\gamma_1})\beta^*(\gamma_2,\nu)]+ \\ &+\sum \mu(\nu_1,\nu_2)(n_{\nu_1}-n_{\nu_2})[K(\gamma_1,\gamma_2;\nu_1,\nu_2)-K(\gamma_2,\gamma_1;\nu_1,\nu_2)]\Bigg\}; \end{aligned} \tag{4}$$

$$\begin{aligned} i\frac{\partial\mu(\gamma_1,\gamma_2)}{\partial t} = \frac{1}{n_{\gamma_1}-n_{\gamma_2}}\Bigg\{ &\sum [\mu(\gamma_1,\nu)(n_{\gamma_1}-n_\nu)\Omega(\gamma_2,\nu) \\ &- \mu(\nu,\gamma_2)(n_\nu-n_{\gamma_2})\Omega(\nu,\gamma_1)] + \sum [\lambda(\nu,\gamma_2)(1-n_\nu-n_{\gamma_2})\beta(\gamma_1,\nu) \\ &+ \lambda^*(\gamma_1,\nu)(1-n_{\gamma_1}-n_\nu)\beta^*(\gamma_2,\nu)] \\ &+\sum \mu(\nu_1,\nu_2)(n_{\nu_1}-n_{\nu_2})[L(\gamma_1,\gamma_2;\nu_1,\nu_2)-L^*(\gamma_2,\gamma_1;\nu_2,\nu_1)] \\ &+\sum [\lambda(\nu_1,\nu_2)(1-n_{\nu_1}-n_{\nu_2})M(\gamma_1,\gamma_2;\nu_1,\nu_2) \\ &- \lambda^*(\nu_1,\nu_2)(1-n_{\nu_1}-n_{\nu_2})M^*(\gamma_2,\gamma_1;\nu_1,\nu_2)]\Bigg\} \end{aligned} \tag{5}$$

and the equations complex-conjugate to (4) and (5). Here

$$\begin{aligned} \Omega(\gamma,\nu) = &\sum E(f_1,f)\,[u^*_{f_1\gamma}u_{f\nu} - v^*_{f_1\nu}v_{f\gamma}] \\ &+\sum\{J(f'',f';f_1,f)\Phi^*(f',f'')v_{f_1\gamma}u_{f\nu} + J(f_1,f;f'',f')\Phi(f'',f')u^*_{f_1\gamma}v^*_{f\nu}\}; \end{aligned}$$

$$\beta(\gamma,\nu) = \sum E(f_1,f)\,[v^*_{f_1\gamma}u_{f\nu} - v^*_{f_1\nu}u_{f\gamma}] \\ + \sum\{J(f'',f';f_1,f)\Phi^*(f',f'')u_{f_1\gamma}u_{f\nu} + J(f,f_1;f'',f')\Phi(f',f'')v^*_{f_1\gamma}v^*_{f\nu}\};$$

$$Z(\gamma_1,\gamma_2;\nu_1,\nu_2) = \sum\Big\{[v_{f_1\gamma_2}F(f_1,f) + u^*_{f_1\gamma_2}\Phi(f_1,f)] \\ \times[2J(f_2,f'';f,f')u^*_{f_2\gamma_1}u_{f'\nu_2}v^*_{f''\nu_1} + J(f'',f';f_2,f)v_{f_2\gamma_1}v^*_{f'\nu_1}v^*_{f''\nu_2}] \\ + [u^*_{f_1\gamma_2}F(f,f_1) + v_{f_1\gamma_2}\Phi^*(f_1,f)] \\ \times[2J(f,f';f'',f_2)v_{f_2\gamma_1}u_{f'\nu_1}v^*_{f''\nu_2} + J(f,f_2;f'',f')u^*_{f_2\gamma_1}u_{f'\nu_1}u_{f''\nu_2}]\Big\} \\ + \frac{1}{2}\sum J(f_1,f_2;f'',f')[u^*_{f_1\gamma_1}u^*_{f_2\gamma_2}u_{f'\nu_1}u_{f''\nu_2} + v_{f'\gamma_1}v_{f''\gamma_2}v^*_{f_2\nu_2}v^*_{f_1\nu_1}];$$

$$Y(\gamma_1,\gamma_2;\nu_1,\nu_2) = \sum\{[v_{f_1\gamma_2}F(f_1,f) + u^*_{f_1\gamma_2}\Phi(f_1,f)][2J(f_2,f'';f,f')u^*_{f_2\gamma_1}v_{f'\nu_1}u^*_{f''\nu_2} \\ + J(f'',f';f,f_2)v_{f_2\gamma_1}u^*_{f'\nu_1}u^*_{f''\nu_2}] + [u^*_{f_1\gamma_2}F(f,f_1) \\ + v_{f_1\gamma_2}\Phi^*(f_1,f)][2J(f,f';f'',f_2)v_{f_2\gamma_1}u^*_{f'\nu_1}v_{f''\nu_2} + J(f_2,f;f'',f')u^*_{f_2\gamma_1}v_{f'\nu_1}v_{f''\nu_2}]\} \\ + \frac{1}{2}\sum J(f'',f';f_1,f_2)[u^*_{f'\nu_1}u^*_{f''\nu_2}v_{f_2\nu_1}v_{f_1\nu_2} + v_{f_1\nu_1}v_{f_2\nu_2}u^*_{f'\nu_1}u^*_{f''\nu_2}],$$

$$K(\gamma_1,\gamma_2;\nu_1,\nu_2) = \sum\{[v_{f_1\gamma_1}F(f_1,f) + u^*_{f_1\gamma_1}\Phi(f_1,f)][2J(f_2,f'';f',f)(u^*_{f''\nu_1}u_{f'\nu_2} \\ - v^*_{f''\nu_2}v_{f'\nu_1})u^*_{f_2\gamma_2} + J(f'',f';f_2,f)(u^*_{f''\nu_1}v^*_{f'\nu_2} - u^*_{f'\nu_1}v^*_{f''\nu_2})v_{f_2\gamma_2}] \\ + [u^*_{f_1\gamma_1}F(f,f_1) + v_{f_1\gamma_1}\Phi^*(f_1,f)][2J(f',f;f_2,f'')(u^*_{f'\nu_1}u_{f''\nu_2} \\ - v^*_{f'\nu_2}v_{f''\nu_1})v_{f_2\gamma_2} + J(f_2,f;f'',f')(u_{f''\nu_2}v_{f'\nu_1} - u_{f'\nu_2}v_{f''\nu_1})u^*_{f_2\gamma_2}]\} \\ + \sum J(f_1,f_2;f'',f')[u^*_{f_1\gamma_2}u^*_{f_1\gamma_2}u_{f'\nu_2}v_{f''\nu_1} + u^*_{f_2\nu_1}v^*_{f_1\nu_2}v_{f'\gamma_1}v_{f''\gamma_2}];$$

$$L(\gamma_1,\gamma_2;\nu_1,\nu_2) = \sum\{[u_{f_1\gamma_1}F(f_1,f) + v^*_{f_1\gamma_1}\Phi(f_1,f)][2J(f_2,f'';f',f)(u^*_{f''\nu_1}u_{f'\nu_2} \\ - v^*_{f''\nu_2}v_{f'\nu_1})u^*_{f_2\gamma_2} + J(f'',f';f,f_2)(u^*_{f''\nu_1}v^*_{f'\nu_2} - u^*_{f'\nu_1}v^*_{f''\nu_2})v_{f_2\gamma_2}] \\ + [v^*_{f_1\gamma_1}F(f,f_1) + u_{f_1\gamma_1}\Phi^*(f_1,f)][2J(f',f;f_2,f'')(u^*_{f'\nu_1}u_{f''\nu_2} \\ - v^*_{f'\nu_2}v_{f''\nu_1})v_{f_2\gamma_2} + J(f,f_2;f'',f')(u_{f'\nu_2}v_{f''\nu_1} - u_{f''\nu_2}v_{f'\nu_1})u^*_{f_2\gamma_2}]\} \\ + \frac{1}{2}\sum J(f_1,f_2;f'',f')[u^*_{f_2\gamma_2}v^*_{f_1\gamma_1}v_{f'\nu_1}u_{f''\nu_2} - u^*_{f''\gamma_2}v^*_{f'\gamma_1}u_{f_1\nu_2}v_{f_2\nu_1}];$$

$$
\begin{aligned}
M(\gamma_1,\gamma_2;\nu_1,\nu_2) = \sum \{ & [v^*_{f_1\gamma_1}F(f,f_1) + u_{f_1\gamma_1}\Phi^*(f_1,f)][2J(f,f';f_2,f'')v_{f_2\gamma_2}u^*_{f'\nu_1}v^*_{f''\nu_2} \\
& + J(f_2,f;f'',f')u^*_{f_2\gamma_2}u_{f'\nu_1}u_{f''\nu_2}] + [u_{f_1\gamma_1}F(f_1,f) \\
& + v^*_{f_1\gamma_1}\Phi(f_1,f)][2J(f_2,f'';f',f)u^*_{f_2\gamma_2}v^*_{f'\nu_1}u_{f''\nu_2} + J(f'',f';f,f_2)v^*_{f_2\gamma_2}v^*_{f'\nu_1}v^*_{f''\nu_2}] \\
+ & [v_{f_1\gamma_2}F(f_1,f) + u^*_{f_1\gamma_2}\Phi(f_1,f)][2J(f_2,f'';f,f')v^*_{f_2\gamma_1}u_{f'\nu_2}v^*_{f''\nu_1} \\
& + J(f'',f';f_2,f)u_{f_2\gamma_1}v^*_{f'\nu_1}v^*_{f''\nu_2}] + [u^*_{f_1\gamma_2}F(f,f_1) \\
& + v_{f_1\gamma_2}\Phi^*(f_1,f)][2J(f',f;f_2,f'')u_{f_2\gamma_1}u_{f'\nu_1}v^*_{f''\nu_2} + J(f,f_2;f'',f')v^*_{f_2\gamma_1}u_{f'\nu_1}u_{f''\nu_2}]\} \\
+ & \sum J(f_1,f_2;f'',f')[u^*_{f_2\gamma_2}v^*_{f_1\gamma_1}u_{f'\nu_1}u_{f''\nu_2} - v^*_{f'\gamma_2}u_{f''\gamma_1}v^*_{f_2\nu_1}v^*_{f_1\nu_2}].
\end{aligned}
$$

We shall solve the obtained system of homogeneous integral equations by representing λ and μ in the form of a sum of normal oscillations

$$
\lambda(\nu_1,\nu_2) = \sum_E e^{-iEt}\xi_E(\nu_1,\nu_2); \qquad \lambda^*(\nu_1,\nu_2) = \sum_E e^{-iEt}\eta_E(\nu_1,\nu_2);
$$

$$
\mu(\nu_1,\nu_2) = \sum_E e^{-iEt}\chi_E(\nu_1,\nu_2). \tag{6}
$$

In the resulting system of secular equations for determining the spectrum, let us pass to the case of a superconducting system of electrons. We note that the spectrum splits into two branches, and in one of them the oscillations occur for particles with opposite spins (in this case $\lambda_{\infty\infty} = 0$), while in the other they occur for particles with identical spins (in this case $\lambda_{-+} = \lambda_{+-} = 0$). The first branch is considered here. For the selected branch of oscillations, separating the spin indices and taking into account the conditions $\lambda(\nu_1,\nu_2) + \lambda(\nu_2,\nu_1) = 0$, $\mu^*(\nu_1,\nu_2) + \mu(\nu_2,\nu_1) = 0$, we obtain a system of six equations. Two of them, corresponding to the amplitudes of normal oscillations μ_{+-} and μ_{-+}, are independent, coincide, and contain a continuous excitation spectrum with a gap. The remaining four equations relate the functions $\xi_{-+}(p_1,p_2)$, $\eta_{+-}(-p_2,-p_1)$, $\chi_{++}(-p_1,p_2)$, and $\chi_{--}(-p_2,p_1)$ at fixed p_1+p_2. Taking $p_1 = p$; $p_2 = -p+q$, they are considered with respect to $\vartheta_q(p) = \xi_{-+}(p_1,p_2) + \eta_{+-}(-p_2,-p_1)$; $\theta_q(p) = \xi_{-+}(p_1,p_2) -$

$-\eta_{-+}(-p_2,p_1)$; $\varkappa_q(p) = \chi_{++}(-p_1,p_2) - \chi_{--}(-p_2,p_1)$; $\pi_q(p) = \chi_{++}(-p_1,p_2) + \chi_{--}(-p_2,p_1)$.

It is not difficult to verify the presence of a continuous spectrum, separated by a gap, if in the equations obtained one chooses the solution $\sim \delta(p-p_\theta)$ and discards terms of order $1/V$:

$$
E = \Omega(p_0) + \Omega(p_0 - q); \quad E = \Omega(p_0) - \Omega(p_0 - q).
$$

For $q = 0$ the equation for $\theta_q(p)$ coincides with the basic equation of superconductivity. Using the known solution of the superconductivity equation ([1,2,6]), we shall seek the solution for small q by means of the expansion

$$E = |q|E_1 + \ldots; \quad \theta_q = u_p v_p + |q|\theta_1(p,l) + \ldots; \quad \varkappa_q = \varkappa_0(p,l) + \ldots; \quad \pi_q = |q|\pi_1(p,l) + \ldots; \quad l = q/|q|.$$

Using the solvability condition for inhomogeneous integral equations, one can find the relation determining E_1. Carrying out explicit calculations in the radially symmetric case for a neutral system with interaction potential

$$J\rho(\xi)\rho(\xi), \quad \rho(\xi) = 1 \text{ for } -a < \xi < a \quad \text{and} \quad \rho(\xi) = 0 \text{ for } -a > \xi \text{ and } \xi > a$$

$(a = \hbar\omega/2)$, in particular in the cases of zero and critical temperature, for the velocity of longitudinal waves one obtains, respectively, the values

$$v_F\sqrt{1 - \frac{C}{mv_F^2}}\,/\,\sqrt{3}; \qquad v_F\sqrt{\frac{2}{9}}.$$

If, near the critical point, the solution is sought in powers of the gap,

$$E_1 = CE_{11} + \ldots; \quad \varkappa_0(p,l) = C\varkappa_{01}(p,l) + \ldots,$$

then one can discover one more branch of collective excitations arising at the critical temperature θ_c. The corresponding longitudinal temperature waves near the critical point have the velocity

$$v_F C\sqrt{\frac{1}{3\theta_c}F_1\left(\frac{4}{\hbar\omega} + \frac{4}{4}\int_{-a}^{a}\frac{1}{\xi^2}\,\text{th}^2\,\frac{\xi}{2\theta_c}\,d\xi\right)},$$

where F_1 is the dimensionless interaction constant ([7]). At zero temperature this branch of collective excitations coincides with the kinematic branch of collective excitations.

In conclusion I express my deep gratitude to N. N. Bogolyubov for his constant attention to the work and to Yu. A. Tserkovnikov for a number of valuable discussions.

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Note: Figure translations are in progress. See original paper for figures.