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Abstract

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ON THE SOLVABILITY OF THE NEUMANN PROBLEM

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In the present note conditions are formulated for the solvability of the Neumann problem for the Laplace operator in a generalized setting.

1°. The Neumann problem with a nonhomogeneous boundary condition. Let Ω be a bounded domain in n-dimensional Euclidean space R_n with boundary Γ. Find an element $u(x) \in \mathring{L}_2^{(1)}(\Omega)$ *, for which the relation

$$\int_\Omega \nabla v \nabla u \, dm_n(x) = \int_\Gamma v\mu\,(dx), \tag{1}$$

is satisfied, where $v(x)$ is an arbitrary function from $C^{(1)}(\overline{\Omega})$; m_n is n-dimensional Lebesgue measure and $\mu(E)$ is a given completely additive function of Borel sets $E \subset \Gamma$, satisfying the condition $\mu(\Gamma) = 0$.

It is natural to call the function μ the flux of $u(x)$ through Γ.

Theorem 1. *For the solvability of the generalized Neumann problem (1) it is necessary and sufficient that the flux μ satisfy the condition*

$$\sup_{v \in C^{(1)}(\overline{\Omega})} \int_0^\infty \frac{\mu^2(E_t)}{\int_{S_t} |\nabla v| \, d\mu_{n-1}} \, dt < \infty, \tag{2}$$

where μ_{n-1} is $(n-1)$-dimensional Hausdorff measure; $S_t = \{x;\ v(x) = t\} \cap \Omega$, $E_t = \{x,\ v(x) \geq t\} \cap \Gamma$.

From Theorem 1 one can derive the following sufficient condition for the solvability of problem (1). Following note ([1]), by F we denote an open subset of Ω, by E a subset of F closed in Ω, and by $C_2^{(n)}(K)$ the n-dimensional 2-conductivity of the conductor $K = F \setminus E$.

Theorem 2. *Let $\mu_{n-1}\Gamma < \infty$. If for any conductor $K = F \setminus E$ in Ω, satisfying the condition $\mu_{n-1}(\overline{F} \cap \Gamma) \leq \frac{1}{2}\mu_{n-1}\Gamma$, the inequality*

$$\mu(\overline{E} \cap \Gamma) \leq \eta\,[c_2^{(n)}(K)]$$

holds, where the function $\eta(t)$ *is nondecreasing and*

$$\int_0^\infty \eta(t)\,\frac{dt}{t^2} < \infty,$$

then the solution of problem (1) exists and is unique.

Remark. If Ω is the unit circle, then the necessary and sufficient condition for the solvability of problem (1) takes the simple form

$$\iint_{\Gamma\Gamma} \ln \frac{1}{|x_1 - x_2|}\,\mu(dx_1)\mu(dx_2) < \infty. \tag{3}$$

If

$$\mu(E) = \int_E \varphi(s)\,ds,$$

where s is the arclength on Γ and $\varphi \in L(\Gamma)$, then condition (3) can be rewritten in the following form:

$$\iint_{\Gamma\Gamma} \left(\int_{x_1}^{x_2} \varphi(s)\,ds\right)^2 \frac{ds(x_1)\,ds(x_2)}{|x_1 - x_2|^2} < \infty.$$

* Here and below we use the notation of note ([1]).

It is easy to show that the last inequality is necessary and sufficient for the solvability of the Neumann problem (1) in a plane domain with smooth boundary, if the modulus of continuity $\omega(t)$ of the angle between the tangent to Γ and a fixed direction satisfies the condition

$$\int_0 \frac{\omega(t)}{t}\,dt < \infty.$$

Definition 1. Let $\mu_{n-1}\Gamma < \infty$. The boundary of the domain Ω belongs to the class $I^{(n-1)}_{2,\nu(t)}$ if there exists a function $\nu(t)$ such that

$$\sup \frac{\nu(\mu_{n-1}(\overline{E} \cap \Gamma))}{c_2^{(n)}(K)} < \infty,$$

where the supremum is taken over conductors $K = F \setminus E$ in Ω such that $\mu_{n-1}(\overline{F} \cap \Gamma) \le \frac{1}{2}\mu_{n-1}\Gamma$. In the case $\nu(t) = t^{2/q^*}$ $(q^* > 0)$ we denote the class $I^{(n-1)}_{2,\nu(t)}$ by $I^{(n-1)}_{2,q^*}$.

Fig. 1

In the following theorem we shall consider problem (1) under the assumption

Fig. 1

Figure 1: Fig. 1

$$\mu(E) = \int_E \varphi \, d\mu_{n-1}.$$

Theorem 3. 1) For the solvability of problem (1) for every $\varphi \in L_q(\Gamma)$

$$\left(2\frac{n-1}{n} \leq q \leq 2, \text{ if } n > 2; \quad 1 < q \leq 2, \text{ if } n = 2\right)$$

it is necessary and sufficient that the boundary of the domain Ω belong to the class

$$I^{(n-1)}_{2,\frac{q}{q-1}}.$$

2) If

$$\Gamma \in I^{(n-1)}_{2,\nu(t)},$$

where $\nu(t)$ is a nondecreasing absolutely continuous function satisfying the condition

$$\int_0 \frac{dt}{[\nu'(t)]^{\frac{q}{q-2}}} < \infty \qquad (q > 2),$$

and if $\varphi \in L_q(\Gamma)$, then problem (1) has a unique solution.

Example 1. It can be shown that the boundary of the domain

$$\Omega_1 : \left\{ \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta}, \ 0 < x_n < 1 \right\}$$

($\beta > 1$) belongs to the class $I^{(n-1)}_{2,q^*}$, where

$$q^* = \frac{\beta(n-2)+1}{\beta(n-1)-1}.$$

Consequently, the largest value of β for which problem (1) is solvable for all functions $\varphi \in L_2(\Gamma)$ orthogonal to one on Γ is equal to two.

Example 2. For the domain Ω_2 shown in Fig. 1 (see (2)), problem (1) is solvable for all $\varphi \in L_2(\Gamma)$ if $\alpha \leq 2$, and is not always solvable if $\alpha > 2$.

2°. The Neumann problem with homogeneous boundary condition

Let Ω be an arbitrary open set in R_n, $\lambda = \text{const} > 0$, and $f(x) \in L_q(\Omega)$. Find a function $u(x) \in W_2^{(1)}(\Omega)$ satisfying the condition

$$\int_\Omega \{(\nabla u \, \nabla v + \lambda uv) \, dm_n(x)\} = \int_\Omega fv \, dm_n(x), \tag{4}$$

where v is any function from $W_2^{(1)}(\Omega)$.

Definition 2. The set Ω belongs to the class $I_{2,q^*}^{(n)}$ $(q^* > 0)$ if there exists a positive constant $M < m_n\Omega$ such that

$$\sup \frac{m_n^{2/q^*} E}{c_2^{(n)}(K)} = \mathfrak{B}(M) < \infty,$$

where the supremum is taken over all conductors $K = F \setminus E$ in Ω satisfying the condition $m_n F \leq M$.

Theorem 4. For the unique solvability of problem (4) for all $f(x) \in L_q(\Omega)$ $\left(\frac{2n}{n+2} \leq q < 2, \text{ if } n > 2; \ 1 < q < 2, \text{ if } n = 2\right)$, it is necessary and sufficient that $\Omega \in I_{2,\frac{q}{q-1}}^{(n)}$.

Theorem 4 follows easily from the results of note ([3]).

Example 3. It was noted in ([2]) that the domain Ω_1 belongs to the class $I^{(n)}_{2,\,2\frac{\beta(n-1)+1}{\beta(n-1)-1}}$. Hence the smallest exponent q for which problem (4) is solvable for any function $f(x) \in L_q(\Omega_1)$ is equal to $2\frac{\beta(n-1)+1}{\beta(n-1)+3}$.

In the paper ([4]) Deny and Lions showed that problem (4) is always solvable in $W_2^{(1)}(\Omega)$ if $f(x) \in L_2(\Omega)$. In the same work it is proved that solvability in $L_2^{(1)}(\Omega)$ of the problem

$$\int_\Omega \nabla u \, \nabla v \, dm_n(x) = \int_\Omega fv \, dm_n(x), \tag{5}$$

where Ω is a domain of finite measure, $f(x)$ is any function in $L_2(\Omega)$ orthogonal to unity, and $v(x) \in W_2^{(1)}(\Omega)$, is equivalent to the validity of the Poincaré inequality

$$\|u\|^2_{L_2(\Omega)} - \frac{1}{m_n\Omega}\left(\int_\Omega u\,dm_n(x)\right)^2 \le c\|\nabla u\|^2_{L_2(\Omega)}.$$

By virtue of one of the theorems of note ([3]), the last inequality is true if and only if $\Omega \in I^{(n)}_{2,2}$.

Example 4. Problem (5) for the domain

$$\Omega_3 : \left\{0 < x_n < \infty,\ \sum_{i=1}^{n-1} x_i^2 \le e^{-\frac{2x_n}{n-1}}\right\}$$

is solvable for all $f(x) \in L_2(\Omega_3)$ orthogonal to unity in Ω_3.

In conclusion we give a criterion for the discreteness of the spectrum of the Neumann problem for the Laplace operator.

Theorem 5. For the discreteness of the spectrum of the Neumann problem in $L_2(\Omega)$, it is necessary and sufficient that $\Omega \in I^{(n)}_{2,2}$ and $\mathfrak{B}(M) \to 0$ as $M \to 0$.

Example 5. For the domain Ω_2 (Fig. 1), the largest value of α for which problem (5) is solvable for any function $f(x) \in L_2(\Omega_2)$ orthogonal to unity is equal to three. For $\alpha < 3$ the spectrum of the Neumann problem is discrete.

All the results of the work carry over, without substantial changes, to equations with variable coefficients.

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References

[1] V. G. Maz' ya, DAN, **140**, No. 2 (1961).
[2] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, **2**, 1951.
[3] V. G. Maz' ya, DAN, **133**, No. 3 (1960).
[4] J. Deny, J. L. Lions, Ann. de l' Inst. Fourier, **5** (1953–1954).

Note: Figure translations are in progress. See original paper for figures.