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Abstract

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MATHEMATICS

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GENERAL QUASILINEAR EQUATIONS OF SECOND ORDER AND SOME CLASSES OF SYSTEMS OF EQUATIONS OF ELLIPTIC TYPE

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1. We consider the quasilinear elliptic equation

$$a_{ij}(x,u,u_{x_k})u_{x_ix_j} + a(x,u,u_{x_k}) = 0 \tag{1}$$

in a bounded domain Ω of variation of the independent variables $x_1, x_2, \dots, x_n$, $n \geqslant 2$.

For equations having a divergent principal part

$$\frac{\partial}{\partial x_i}(a_i(x,u,u_{x_k})) + a(x,u,u_{x_k}) = 0, \tag{2}$$

in the papers ([1, 2a, ,]) all estimates necessary for proving the existence of classical (twice continuously differentiable) solutions of the first boundary-value problem were obtained: successively there are estimated $\max_\Omega |u|$, $|u|_{C_{0,\alpha}(\Omega)}$, $\max_\Omega |u_{x_k}|$, $|u_{x_k}|_{C_{0,\alpha}(\Omega)}$.

In the general case equation (1) cannot be represented in the form (2). For solutions $u(x_1, x_2)$ of equations (1) with two independent variables there are known the theorems of S. N. Bernstein ([3]) and L. Nirenberg ([4]), which make it possible to estimate $|u_{x_k}|_{C_{0,\alpha}(\Omega)}$ through $\max_\Omega |u_{x_k}|$, $k = 1, \dots, n$.

In the present paper this result is generalized to the case of equations (1) with arbitrary $n \geqslant 2$.

2. Theorem 1. Let $u(x)$ belong to $W_2^2(\Omega)$, satisfy equation (1) almost everywhere in Ω, and let $\operatorname{vrai}\max_\Omega |u| \leqslant M$, $\operatorname{vrai}\max_\Omega |u_{x_k}| \leqslant M_1$, $k = 1, \dots, n$. Suppose, moreover, that for $x \in \Omega$, $|u| \leqslant M$, $p_k \leqslant M_1$, the inequalities

$$a_{ij}\xi_i\xi_j \geqslant \nu \sum \xi_i^2,$$

$$|a_{ij}|, \quad \left|\frac{\partial a_{ij}}{\partial p_k}\right|, \quad \left|\frac{\partial a_{ij}}{\partial u}\right|, \quad \left|\frac{\partial a_{ij}}{\partial x_k}\right|, \quad |a| \leqslant \mu \tag{3}$$

hold for $a_{ij}(x, u, p_k)$ and $a(x, u, p_k)$, with positive constants ν and μ. Then $u \in C_{1,\alpha}(\Omega')$, where $\alpha > 0$, $|u_{x_k}|_{C_{0,\alpha}(\Omega')}$, $\Omega' \Subset \Omega$, is determined only by the quantities ν, μ, n, M_1 and by the distance of Ω' to the boundary S of the domain Ω.

If S belongs to C_2 (see ([2a])) and $u|_S \in C_{1,\beta}(S)$, then $|u_{x_k}|_{C_{0,\alpha}(\Omega)}$ is estimated in terms of ν, μ, n, M_1, the norm of S in C_2, and $|u|_{C_{1,\beta}(S)}$, $\beta \geqslant \alpha$. To prove this theorem, the method developed in ([5,1,2a,]) is sharpened.

We explain the proof of the first part of the theorem.

Consider the $2n$ functions

$$w_\pm^r = (\operatorname{grad} u)^2 + 5n(1 \pm M_1^{-1} u_{x_r}), \qquad r = 1, \dots, n. \tag{4}$$

Each of these functions satisfies the integral identity

$$\int_\Omega \left(a_{ij} u_{x_k x j} u_{x_k x i} \eta + \frac{1}{2} a_{ij} w^r_{x_j} \eta_{x_i} + \frac{1}{2} b^l_{ij} u_{x_l x j} w^r_{x_i} \eta + c^r_{ij} u_{x_i x j} \eta + d^r_i \eta_{x_i} + f^r \eta \right) dx = 0. \tag{5}$$

Here $b^l_{ij}, c^r_{ij}, d^r_i, f^r$ are expressed in terms of u_{x_k}, a, and the derivatives of a_{ij} with respect to the arguments, while $\eta(x)$ is an arbitrary bounded function from $\overset{0}{W}{}_2^1(\Omega)$.

We shall denote by $A^{r\pm}_{\lambda,\rho}$ the set of points x of the ball $K(\rho) \subset \Omega$ for which $w^r_\pm(x) > \lambda$, and by $B^{r\pm}_{\lambda,\rho}$ the set of points in $K(\rho)$ where $w^r_\pm(x) < \lambda$.

From (5) it is established that $w^r_\pm$ obey the inequalities

$$\int_{A^{r\pm}_{\lambda,\rho}} (\operatorname{grad} w^r_\pm)^2 \xi^2 \, dx \leq \gamma \int_{A^{r\pm}_{\lambda,\rho}} (w^r_\pm - \lambda)^2 (\operatorname{grad} \xi)^2 \, dx + \gamma \operatorname{mes} A^{r\pm}_{\lambda,\rho}. \tag{6}$$

for an arbitrary bounded nonnegative $\xi(x)$ from $\overset{0}{W}{}_2^1(\Omega)$ and for $\lambda \geq \max_{K(\rho)} w_{\pm}^r - \delta$, where $\delta > 0$ and γ are constants depending only on ν and μ from (3).

These inequalities proved sufficient for estimating $|u_{x_k}|_{C_{0,\alpha}(\Omega')}$, $k = 1, \ldots, n$, for some $\alpha > 0$.

Theorem 1, together with the previously obtained estimates ([6,2a]) of the maxima of the moduli u_{x_k}, makes it possible to prove the following theorems.

Theorem 2. Let $u(x)$ be a solution of equation (1) from $C_3(\Omega)$, and let $a_{ij}(x, v, p_k)$ and $a(x, v, p_k)$, for $x \in \Omega$, $|v| \leq \max_\Omega |u(x)| = M$, and $|p_k| < \infty$, satisfy the natural compatibility conditions on the orders of growth with respect to $p = \left(\sum_{k=1}^n p_k^2\right)^{1/2}$ as $p \to \infty$, namely:

$$a_{ij}(x, v, p_k)\xi_i\xi_j \geq \nu_1 \sum \xi_i^2, \qquad \nu_1 > 0,$$

$$|a|+\left|\frac{\partial a}{\partial v}\right|+\left|\frac{\partial a}{\partial x_k}\right|+(1+p)\left|\frac{\partial a}{\partial p_k}\right|+(1+p^2)\left(|a_{ij}|+\left|\frac{\partial a_{ij}}{\partial v}\right|+\left|\frac{\partial a_{ij}}{\partial x_k}\right|\right)+(1+p)^3\left|\frac{\partial a_{ij}}{\partial p_k}\right| \leq \mu_1(1+p)^2. \tag{7}$$

Then $|u|_{C_{1,\alpha}(\Omega')}$ and $\alpha > 0$ are determined by the quantities $\nu_1, \mu_1, n, |u|_{C_{0,\beta}(\Omega)}$, $\beta > 0$, and by the distance from Ω' to S.

If $S \in C_2$ and $u|_S \in C_{1,\alpha}(S)$, then $|u|_{C_{1,\alpha}(\Omega)}$ and α depend only on $\nu_1, \mu_1, n, |u|_{C_{0,\beta}(\Omega)}$, the norm of S in C_2, and $|u|_{C_{1,\alpha}(S)}$.

Remark. We have somewhat modified the method of obtaining the estimates $\max_\Omega |u_{x_k}|$ given in ([6,2a]). This makes it possible to lower the smoothness requirements on u, so that in Theorems 2 and 3 u may be taken from $C_1(\Omega) \cap \overset{*}{W}{}_2^2(\Omega)$.

Theorem 3. Suppose that, in addition to all the requirements of Theorem 2, the conditions of "weakened occurrence of v in a_{ij} and a" are fulfilled:

$$1) \quad \left|\frac{\partial a_{ij}(x, v, p_k)}{\partial v}\right| \leq \varepsilon;$$

$$2) \quad \left|\frac{\partial a(x, v, p_k)}{\partial v}\right| \leq \varepsilon p^2 + P(p) \quad \text{or} \quad \frac{\partial a(x, v, p_k)}{\partial v} \leq 0,$$

where ε is a small con-

constant, and $p^2 P(p) \to 0$ as $p \to \infty$. Then in all the assertions of the preceding theorem $|u|_{C_{0,\beta}(\Omega)}$ may be replaced by $M = \max_\Omega |u|$.

Let us formulate one of the existence theorems following from the estimates obtained and from Schauder estimates for solutions of linear equations.

Theorem 4. Let $S \in C_{2,\alpha}$, $\varphi(s) \in C_{2,\alpha}(S)$, and let the functions $a_{ij}(x, v, p_k)$ and $a(x, v, p_k)$, for $x \in \Omega$ and arbitrary v, p_k, have derivatives with respect to their arguments that are bounded for any finite v, p_k, and also satisfy inequalities (7) and conditions 1), 2) of Theorem 3. If, for solutions $u(x, \tau)$ of the boundary-value problems

$$L_\tau u \equiv \tau L u + (1-\tau)\Delta u = 0, \qquad u|_S = \tau\varphi(s), \qquad \tau \in [0,1],$$

one can give an estimate $\max_\Omega |u(x,\tau)|$ uniformly with respect to $\tau \in [0,1]$, then there exists a solution $u(x) \in C_{2,\alpha}(\Omega)$ of the boundary-value problem $Lu = 0, \quad u|_S = \varphi(s)$.

3. We considered the first boundary-value problem for elliptic systems of quasilinear equations of the form

$$Mu \equiv a_{ij}(x, \mathbf{u})\, \mathbf{u}_{x_i x_j} + b_i(x, \mathbf{u}, \mathbf{u}_{x_k})\, \mathbf{u}_{x_i} + \mathbf{a}(x, \mathbf{u}, \mathbf{u}_{x_k}) = 0, \tag{8}$$

where $\mathbf{u} = (u^1, \dots, u^N)$,

$$a_{ij}(x, \mathbf{u})\xi_i\xi_j \geq \nu(|\mathbf{u}|)\sum \xi_i^2,$$

$$|b_i(x, \mathbf{u}, \mathbf{p}_k)| \leq \mu(|\mathbf{u}|)(1+p), \qquad p = \left[\sum_{l=1}^{N}\sum_{k=1}^{n}(p_k^l)^2\right]^{1/2}, \tag{9}$$

$$|\mathbf{a}(x, \mathbf{u}, \mathbf{p}_k)| \leq \mu(|\mathbf{u}|)\varepsilon p^2 + P(p), \qquad \varepsilon \ll 1, \qquad p^{-2}P(p) \to 0, \quad p \to \infty,$$

$\nu(t)$ and $\mu(t)$ are positive continuous functions, $t \in [0,\infty)$. Among the works in the literature on systems of quasilinear elliptic equations with $n > 2$, we point to the work of M. I. Višik ([7]) on generalized solutions of systems of a more general form, but such that, in a certain sense, the methods of studying linear equations are applicable to them.

In addition to conditions (9), we shall make the following assumptions on the functions $a_{ij}(x, \mathbf{u})$, $b_i(x, \mathbf{u}, \mathbf{p}_k)$, $a^l(x, \mathbf{u}, \mathbf{p}_k)$: for $x \in \Omega$ and finite $|\mathbf{u}|, |\mathbf{p}_k|$, the functions a_{ij} have bounded derivatives, while b_i and a^l satisfy a Hölder condition with respect to their arguments. Let $S \in C_{2,\alpha}$, $\vec{\varphi}(s) \in C_{2,\alpha}(S)$.

Theorem 5. If the conditions just stated are satisfied and if one can give an estimate $\max_\Omega |\mathbf{u}(x,\tau)|$ of the solutions $\mathbf{u}(x,\tau)$ of the problem

$$M_\tau \mathbf{u} \equiv M\mathbf{u} +$$

$$+(1-\tau)\Delta\mathbf{u}, \qquad \mathbf{u}|_S = \tau\vec{\varphi}(s),$$

uniformly with respect to $\tau \in [0,1]$, then there exists a solution $\mathbf{u} \in C_{2,\alpha}(\Omega)$ of the boundary-value problem $M\mathbf{u} = 0$, $\mathbf{u}|_S = \vec{\varphi}(s)$.

4. For systems of linear equations with unbounded coefficients

$$\frac{\partial}{\partial x_i}(a_{ij}(x)u_{x_j} + A_i(x)u + \mathbf{f}_i(x)) + B_i(x)u_{x_i} + B(x)u + \mathbf{a}(x) = 0, \quad (10)$$

where A_i, B_i, B are matrices with elements a_i^{lm}, b_i^{lm}, b^{lm}, respectively, $\mathbf{u} = (u^1, \dots, u^N)$, $\mathbf{f}_i = (f_i^1, \dots, f_i^N)$, $\mathbf{a} = (a^1, \dots, a^N)$, we investigated the question of smoothness of generalized solutions. Along this path we obtained all the same results as were established for the case of a single equation in works ([8],[9],[2r]). Namely, the following holds.

Theorem 6. Suppose the following conditions are satisfied:

$$a_{ij}\xi_i\xi_j \geq \nu \sum \xi_i^2, \qquad x \in \Omega, \qquad \nu > 0, \quad (11)$$

$$\|a_i^{lm}, b_i^{lm}, f_i^l\|_{L_q(\Omega)} + \|b^{lm}, a^l\|_{L_{\frac{q}{2}}(\Omega)} + \max_\Omega |a_{ij}| \leq \mu, \qquad q > n.$$

Then every solution $\mathbf{u}(x)$ from $W_2^1\Omega$ of system (10) satisfies a Hölder condition in $\Omega' \subset \Omega$, with exponent $\alpha > 0$, and $|\mathbf{u}|_{C_{0,\alpha}(\Omega')}$ depends only on ν, μ and q from (11), on n, on $|\mathbf{u}|_{L_2(\Omega)}$, and on the distance from Ω' to S. If $\mathbf{u}|_S \in C_{0,\alpha}(S)$, then, under small assumptions on S (condition (A) from ([2a])), $|\mathbf{u}|_{C_{0,\alpha}(\Omega)}$ is bounded.

Let us note that the conditions (11) cannot be weakened, as is evident from the examples constructed in ([2r]).

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Note: Figure translations are in progress. See original paper for figures.