



---

Soviet-era science, translated into English

# D. V. ANOSOV

1962

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.53553>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**D. V. ANOSOV**

## **STRUCTURAL STABILITY OF GEODESIC FLOWS ON COMPACT RIEMANNIAN MANIFOLDS OF NEGATIVE CURVATURE**

*(Presented by Academician L. S. Pontryagin, 19 III 1962)*

1. The specification on a compact manifold  $W^m$  of a smooth vector field  $f(w)$  determines a dynamical system

$$\dot{w} = f(w). \quad (1)$$

This dynamical system is called structurally stable if, for every vector field  $g(w)$  sufficiently close to  $f(w)$  in the  $C^1$  sense, there exists a homeomorphism (in general, not smooth)  $\chi : W^m \rightarrow W^m$ , close to the identity in the  $C^0$  sense and carrying trajectories of the system (1) into trajectories of the perturbed system

$$\dot{w} = g(w). \quad (2)$$

2. Let  $V^n$  be a compact Riemannian manifold of negative curvature (the curvature must be negative at every point and in every two-dimensional direction), and let  $W^{2n-1}$  be the space of line elements of  $V^n$ . The geodesic lines of the manifold  $V^n$  determine a certain dynamical system ("geodesic flow") in  $W^{2n-1}$ . I shall prove Smale's conjecture that this dynamical system is structurally stable. (In this connection see <sup>(3,4)</sup>.)

The proof consists in the fact that, if the curvature is negative, then the geodesic flow satisfies certain "conditions" ( $Y$ ), which are formulated below and which turn out to be sufficient for structural stability. The fact that the conditions ( $Y$ ) are fulfilled for the geodesic flow was in fact proved by Hadamard and É. Cartan (see Appendix III to the book <sup>(1)</sup>); the main idea of the proof that these conditions are sufficient for structural stability will be indicated in § 5.

3. The system (1) determines a group of transformations  $F_t : W^m \rightarrow W^m$  ( $F_t(w)$  is the value at time  $t$  of the solution passing through the point  $w$  at  $t = 0$ ). The  $F_t$  induce the corresponding mappings of the tangent bundle:

$$\tilde{F}_t : T(W^m) \rightarrow T(W^m).$$

The system of variational equations for the system (1) describes the dynamical system  $\{\tilde{F}_t\}$  in  $T(W^m)$  and has the form (in the usual notation)

$$\dot{\omega} = f_w(F_t(w))\omega. \quad (3_w)$$

Here  $\omega(t) = \tilde{F}_t\omega(0) \in R_{F_t(w)}^m$ .  $\omega$  is a vector, but neither  $\dot{\omega}$  nor  $f_w$  has tensor character.

\* $R_w^m$  denotes the tangent space to  $W^m$  at the point  $w$ .

The conditions ( ) are as follows:

- ( 1).  $f(w) \neq 0$  for all  $w$ .
- ( 2). Every  $R_w^m$  decomposes into a direct sum

$$R_w^m = X_w^k \oplus Y_w^l \oplus R_w^1, \quad \dim X = k \neq 0, \quad \dim Y = l \neq 0,$$

where  $R_w^1$  is generated by the vector  $f(w)$ , and, moreover:

- a) every solution of the system (3<sub>w</sub>) for which  $\omega(0) \in X_w^k$  satisfies the inequalities

$$|\omega(t)| \leq a|\omega(0)|e^{-ct} \quad \text{for } t \geq 0, \quad |\omega(t)| \geq b|\omega(0)|e^{-ct} \quad \text{for } t \leq 0;$$

- b) every solution of the system (3<sub>w</sub>) for which  $\omega(t) \in Y_w^l$  satisfies the inequalities

$$|\omega(t)| \leq a|\omega(0)|e^{ct} \quad \text{for } t \leq 0, \quad |\omega(t)| \geq b|\omega(0)|e^{ct} \quad \text{for } t \geq 0.$$

The constants  $a, b, c$  are positive and the same for all  $w$  and all  $\omega(t)$  with initial values in  $X_w^k \cup Y_w^l$ .

Condition ( 1) means that the system has no equilibrium positions; this is also the case for geodesic flows.  $f(F_t(w))$  is always one of the solutions of (3<sub>w</sub>), and this solution, by compactness and ( 1), tends neither to 0 nor to  $\infty$ .

( 2) describes the behavior of all the remaining solutions. It is easy to show that the subspaces  $X_w^k$  and  $Y_w^l$  are determined uniquely by their properties a) and b), that  $k, l$  are the same for all  $w$ , and that  $X_w^k, Y_w^l$  depend continuously on  $w$  (moreover, it is clear that if  $w$  varies along a trajectory, then they vary smoothly).

- 4. Since we are interested only in the mutual disposition of trajectories, and not in shifts along them, it is expedient to introduce a new coordinate system, in which one coordinate would be measured along the trajectory and would be proportional to time, while the remaining coordinates would be measured along small surfaces  $\Pi(w)$  transverse to the trajectories. Such a system must be introduced along each trajectory, and this must be

done in some sense “consistently” and “uniformly.” Here is a more precise description. At each point  $w \in W^m$  we construct a small smooth  $(m - 1)$ -dimensional surface  $\Pi(w)$ , having at its center  $w$  the tangent plane  $X_w^k \oplus Y_w^l$ ; as  $w$  varies, the surface  $\Pi(w)$  must vary continuously, and when  $w$  varies along a trajectory, even smoothly. Take the trajectory  $F_t(w_0)$  and consider some point  $w_1$  near it. The point  $w_1$  lies on one of the surfaces  $\Pi(F_t(w_0))$ ; let, say,  $w_1 \in \Pi(F_{t_1}(w_0))$ . In order to determine the position of the point  $w_1$ , it is therefore necessary to specify the number  $t_1$  and indicate the position of  $w_1$  on the surface  $\Pi(F_{t_1}(w_0))$ . To describe the behavior of some trajectory near  $F_t(w_0)$ , one must follow how, as  $t$  varies, the point of intersection of the trajectory under consideration with  $\Pi(F_t(w_0))$  changes; and for the motion of this point one obtains a certain system of differential equations. All this must be done for all  $w_0 \in W^m$ , so that the indicated system of differential equations contains  $w_0$  as a parameter.

5. The main idea in proving the sufficiency of conditions ( ) for roughness is the following. We try to find that trajectory of system (2) into which, under the homeomorphism  $\chi$ , the trajectory  $F_t(w)$  of system (1) is transformed. The required trajectory of system (2) must, for all  $t$ —both for  $t \geq 0$  and for  $t \leq 0$ —lie near the original trajectory  $F_t(w)$ . It turns out that if  $g - f$  is sufficiently small, then for any  $w$  those points  $w' \in \Pi(w)$  through which pass trajectories of system (2) that remain, for all  $t \geq 0$ , in some small neighborhood of the original trajectory of system (1), form a smooth  $k$ -dimensional manifold  $M^k(w) \subset \Pi(w)$ , while those points  $w'' \in \Pi(w)$ , through ...

which pass through the trajectories of system (2) that remain, for all  $t \leq 0$ , in a certain small neighborhood of the initial trajectory of system (1), form a smooth  $l$ -dimensional manifold  $N^l(w) \subset \Pi(w)$ . As  $w$  varies,  $M^k(w)$  and  $N^l(w)$  vary continuously. The tangent planes to  $M^k(w)$  are close to  $X_w^k$ , and the tangent planes to  $N^l(w)$  are close to  $Y_w^l$ ; thus  $M^k(w)$  and  $N^l(w)$  intersect in one and only one point  $w' \in \Pi(w)$ , and  $w'$  depends continuously on  $w$ . As is easy to see, the mapping  $\chi : w \rightarrow w'$  gives the desired homeomorphism.

If the assertions formulated in the preceding paragraph are rephrased in terms of the systems discussed in Section 4 (and which describe the behavior of the trajectories of system (2) near fixed trajectories  $F_t(w)$  of system (1)), then one obtains a theorem analogous to the well-known Hadamard-Perron theorem on invariant manifolds (also proved by many other authors, but later), and which may be characterized as “a certain theorem on conditional stability under continuously acting perturbations.” Probably any proof of the Hadamard-Perron theorem can be adapted for our purposes as well. For example, the proof given in (2) is quite suitable (with the appropriate modifications).

*Remark in proof.* One can show that a system satisfying the conditions (V) and possessing an integral invariant is ergodic, and that its periodic trajectories form an everywhere dense set.

Steklov Mathematical Institute  
Academy of Sciences of the USSR

Received  
1 III 1962

## REFERENCES

- <sup>1</sup> É. Cartan, *Geometry of Riemannian Spaces*, Moscow-Leningrad, 1936.
- <sup>2</sup> D. V. Anosov, Scientific Reports of Higher School, Phys.-Math. Sciences, No. 1, 3 (1959).
- <sup>3</sup> S. Smale, Report at the Symposium on Nonlinear Oscillations, Kiev, 1961.
- <sup>4</sup> V. I. Arnold, Ya. G. Sinai, DAN, 144, No. 4, 695 (1962).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*