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Abstract

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MATHEMATICS

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ON THE ANALYTIC ITERATION OF FUNCTIONS OF TWO VARIABLES

(Presented by Academician S. L. Sobolev on 23 X 1961)

The problem considered is that of finding functions $u(x, y, t)$, $v(x, y, t)$, analytic in x, y and continuous in t, satisfying the conditions (see (^{1},^{2})):

1) $u(x, y, 0) = x, \quad v(x, y, 0) = y;$

2) $u(x, y, 1) = u(x, y), \quad v(x, y, 1) = v(x, y);$

3) $u[u(x, y, t_1), v(x, y, t_1), t_2] = u(x, y, t_1 + t_2),$

$$v[u(x, y, t_1), v(x, y, t_1), t_2] = v(x, y, t_1 + t_2).$$

It is assumed that the given functions $u(x, y)$, $v(x, y)$ have a fixed point (α, β) and, throughout the domain E invariant with respect to this transformation, satisfy the inequality

$$\frac{D[u(x, y),\, v(x, y)]}{D(x, y)} > 0. \tag{1}$$

Other assumptions concerning the characteristic roots are formulated in § 1.

§ 1. Let $\alpha = \beta = 0$, and let preliminary formal transformations have been carried out, in consequence of which the given functions $u(x, y)$, $v(x, y)$ are represented at the given point by the series

$$\begin{aligned} u(x, y) &= \rho_1 x + A_{20}x^2 + A_{11}xy + A_{02}y^2 + \cdots, \\ v(x, y) &= \rho_2 y + B_{20}x^2 + B_{11}xy + B_{02}y^2 + \cdots. \end{aligned} \tag{2}$$

It is assumed that

$$\rho_1 \neq \rho_2, \qquad \rho_1 \neq 1, \qquad \rho_2 \neq 1, \qquad \rho_1\rho_2 > 0. \tag{3}$$

We shall seek the functions $u(x, y, t)$, $v(x, y, t)$ in the form of series

$$\begin{aligned} u(x,y,t) &= \rho_1^t x + \alpha_{20}(t)x^2 + \alpha_{11}(t)xy + \alpha_{02}(t)y^2 + \cdots, \\ v(x,y,t) &= \rho_2^t y + \beta_{20}(t)x^2 + \beta_{11}(t)xy + \beta_{02}(t)y^2 + \cdots \end{aligned} \tag{4}$$

with coefficients continuous in t.

Theorem 1. If the formal series (4), satisfying condition 3), exist, then the coefficients $\alpha_{mp}(t)$, $\beta_{mp}(t)$ $(m+p \geq 2)$ are polynomials in the quantities ρ_1^t, ρ_2^t, t with constant coefficients.

From this theorem and conditions 2) and 1), a simple method of successive determination of the coefficients of the series (4) is easily derived.

Concerning the fact of existence of the formal series (4) there is the following theorem:

Theorem 2. If ρ_1, ρ_2 satisfy (3), then there exist formal series (4) satisfying conditions 1), 2), 3).

The proof of this theorem is carried out by means of solving the system of functional equations

$$\begin{aligned} f[u(x,y), v(x,y), z+1] &= \rho_1 f(x,y,z), \\ g[u(x,y), v(x,y), z+1] &= \rho_2 g(x,y,z), \end{aligned} \tag{5}$$

which can be represented in the form of formal power series in x, y with coefficients that are polynomials in z and with functional determinant not equal to zero at $x = y = 0$. The functions $u(x,y,z,t)$, $v(x,y,z,t)$, defined by means of this solution by the relations

$$\begin{aligned} f[u(x,y,z,t),\ v(x,y,z,t),\ z+t] &= \rho_1^t f(x,y,z), \\ g[u(x,y,z,t),\ v(x,y,z,t),\ z+t] &= \rho_2^t g(x,y,z), \end{aligned} \tag{6}$$

do not depend on z. From this fact the assertion of the theorem is easily derived.

On the convergence of the series (4) (or, what is the same, of the solution (5)) one can prove a theorem from which the theorems of the articles $^{(1,2)}$ follow as a special case:

Theorem 3. *Suppose that convergent series (2) are given for which the inequalities (3) hold and*

$$|\rho_1| < 1, \quad |\rho_2| < 1 \quad (\text{or } |\rho_1| > 1, \quad |\rho_2| > 1). \tag{7}$$

Then the system of functional equations (5) has a convergent solution in the form of power series in x, y, in which the coefficients are polynomials in z, and whose functional determinant is not equal to zero at $x = y = 0$. The sizes of the domains of convergence with respect to z do not depend on z.

The functions $u(x,y,t)$, $v(x,y,t)$ determine in the space u,v a phase trajectory passing through the point (x,y). The equation $y=y(x)$ of every phase trajectory satisfies the functional equation

$$y(x)=v_{-1}\{u[x,y(x)],\ y[u(x,y(x))]\}, \tag{8}$$

where $u_{-1}(x,y)$, $v_{-1}(x,y)$ are the functions inverse to $u(x,y)$, $v(x,y)$. For trajectories of the form

$$y(x)=a_2x^2+a_3x^3+\cdots \tag{9}$$

the following theorem has been proved:

Theorem 4. *Suppose that in the series (2) the numbers* ρ_1,ρ_2 *satisfy either* $|\rho_1|<1$, $|\rho_2|<1$, *and at the same time the equalities* $\rho_1=\rho_2^n$ *or* $\rho_2=\rho_1^m$, *where* $n,m\geq 2$, *do not hold, or*

$$0<|\rho_1|<1, \qquad |\rho_2|>1. \tag{10}$$

Then there exists one and only one series (9) which satisfies the functional equation (8).

§ 2. In § 4 of article (1) and in § 6 of article (2) a definition of an iteration domain was given. Bearing in mind, generally speaking, iteration domains of two complex variables, one can prove a theorem on the analytic continuation of the functions $u(x,y,t)$, $v(x,y,t)$ into this domain:

Theorem 5. *Suppose the functions* $u(x,y,t)$, $v(x,y,t)$ *satisfy conditions 1), 2), 3), for* $u(x,y)$, $v(x,y)$ *the inequality (1) holds, and at the point* (α,β) *there are convergent series*

$$\begin{aligned} u(x,y,t)&=\alpha+\alpha_{10}(t)(x-\alpha)+\alpha_{01}(t)(y-\beta)+\cdots,\\ v(x,y,t)&=\beta+\beta_{10}(t)(x-\alpha)+\beta_{01}(t)(y-\beta)+\cdots. \end{aligned} \tag{11}$$

Then in the whole iteration domain corresponding to that neighborhood of the point (α,β) *which is the domain of convergence of the series (11), there exists one and only one pair of functions* $u'(x,y,t)$, $v'(x,y,t)$, *which satisfies conditions 1), 2), 3), is expandable at any point* (x_0,y_0) *of this domain into the series*

$$\begin{aligned} u'(x,y,t)&=\alpha(x_0,y_0,t)+\alpha_{10}(x_0,y_0,t)(x-x_0)+\alpha_{01}(x_0,y_0,t)(y-y_0)+\cdots,\\ v'(x,y,t)&=\beta(x_0,y_0,t)+\beta_{10}(x_0,y_0,t)(x-x_0)+\\ &\quad+\beta_{01}(x_0,y_0,t)(y-y_0)+\cdots \end{aligned} \tag{12}$$

and coincides with $u(x, y, t)$, $v(x, y, t)$ in the domain of convergence of the series (11).

As an example, let us consider the functions $u(x, y)$, $v(x, y)$, which are obtained as the solution at $t = 1$ of the system of equations

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = -\omega^2 \sin u - \gamma v, \tag{13}$$

where $\gamma > 0$, $\omega^2 > \gamma^2/4$; x, y are the values of the solution at $t = 0$. The coordinates of all fixed points are given by the expressions

$$\alpha_k = k\pi, \qquad \beta = 0, \tag{14}$$

where $k = 0, \pm 1, \pm 2, \ldots.$ The Taylor expansions at these points of the right-hand sides of the differential equations for the sought functions $u(x, y, t)$, $v(x, y, t)$, which have the form (11), are as follows:

$$\begin{aligned} \omega(x, y) &= \omega_{10}^{(k)}(x - \alpha_k) + \omega_{01}^{(k)} y + \cdots, \\ \sigma(x, y) &= \sigma_{10}^{(k)}(x - \alpha_k) + \sigma_{01}^{(k)} y + \cdots. \end{aligned} \tag{15}$$

It is not difficult to verify that

$$\begin{aligned} \omega_{10}^{(s)} &= -\frac{\gamma}{2} + ik\pi + \frac{A_{10}^{(s)} - B_{01}^{(s)}}{2e^{-\gamma/2}} \, \frac{D^{(s)} + m\pi}{\sin D^{(s)}}, & \omega_{01}^{(s)} &= \frac{A_{01}^{(s)}}{e^{-\gamma/2}} \, \frac{D^{(s)} + m\pi}{\sin D^{(s)}}, \\ \sigma_{10}^{(s)} &= \frac{B_{10}^{(s)}}{e^{-\gamma/2}} \, \frac{D^{(s)} + m\pi}{\sin D^{(s)}}, & \sigma_{01}^{(s)} &= -\frac{\gamma}{2} + ik\pi - \frac{A_{10}^{(s)} - B_{01}^{(s)}}{2e^{-\gamma/2}} \, \frac{D^{(s)} + m\pi}{\sin D^{(s)}}, \end{aligned} \tag{16}$$

where k, m are arbitrary integers whose sum is even; i is the imaginary unit; $s = 1, 2$; $A_{10}^{(s)}$, $A_{01}^{(s)}$, $B_{10}^{(s)}$, $B_{01}^{(s)}$ are the first coefficients of the Taylor expansions of the functions $u(x, y)$, $v(x, y)$ at the points (14); $D^{(1)} = D'$, $D^{(2)} = i\delta$, while D' and δ are the least positive roots of the equations

$$\cos D^{(s)} = \frac{A_{10}^{(s)} + B_{01}^{(s)}}{2e^{-\gamma/2}} = \cos \sqrt{\pm \omega^2 - \frac{\gamma^2}{4}}, \tag{17}$$

where the plus sign pertains to the case $s = 1$, and the minus sign to the case $s = 2$. In the article [2] it is shown that, if the inequalities

$$\rho_1^{(s)} \neq \rho_1^{(s)n-q} \rho_2^{(s)q}, \qquad \rho_2^{(s)} \neq \rho_1^{(s)n-q} \rho_2^{(s)q},$$

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Fig. 1

Figure 1: Fig. 1

where $n \geq 2$, $0 \leq q \leq n$, are satisfied, then the remaining coefficients of the series (15) will be uniquely determined through (16). It is not difficult to verify that, in the case under consideration, these inequalities for the characteristic roots at the points (14) are satisfied. All coefficients of the series (15) will be real if the coefficients (16) are real.

Fig. 1

For $s = 1$ the expressions (16) will be real if one sets $k = 0$, and m equal to any even number. Consequently, there are infinitely many such solutions. For $s = 2$ the expressions (16) will be real only when $k = m = 0$, i.e. there exists only one real solution. Therefore, according to Theorem 5, by analytic continuation of the solution from the origin of coordinates into a neighborhood of the points $(\pm\pi, 0)$ (the iterational domain is enclosed between the separatrices AB and CD, see Fig. 1) we obtain infinitely many analytic solutions. But since in a neighborhood of each of these points there exists only one real solution of type (11), all these solutions, except one, are singular at the origin of coordinates. In this connection it seems interesting to consider the question of the general solution of the problem of analytic iteration.

3. The general solution of the problem can be obtained starting from the system of functional equations

$$\begin{aligned} H'[u(x,y),\, v(x,y)] &= H(x,y), \\ F[u(x,y),\, v(x,y)] &= F(x,y) + 1. \end{aligned} \tag{18}$$

If $u(x,y,t)$ and $v(x,y,t)$ are found in advance, for example in the form of series (11), then equations (18) can be solved by reducing them to differential equations. Let $H_1(x,y)$ and $F_1(x,y)$ be one of the solutions of (18) obtained in this way. Then the general solution of the system (18) will be

$$\begin{aligned} H(x,y) &= w_1[H_1(x,y),\, F_1(x,y)], \\ F(x,y) &= F_1(x,y) + w_2[H_1(x,y),\, F_1(x,y)], \end{aligned} \tag{19}$$

where w_1 and w_2 are arbitrary functions periodic in the second argument with period equal to unity. Having the functions (19), it is not difficult to construct the right-hand sides of the differential equations which the desired functions $u(x,y,t)$, $v(x,y,t)$ of general form must satisfy (see formulas (1.3) from $^{(2)}$).

As an example let us take the linear case:

$$u(x,y) = A_{10}x + A_{01}y, \qquad v(x,y) = B_{10}x + B_{01}y. \tag{20}$$

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Then

$$H_1(x, y) = \frac{\ln(t_{11}x + t_{12}y)}{\ln \rho_1} - \frac{\ln(t_{21}x + t_{22}y)}{\ln \rho_2}, \qquad F_1(x, y) = \frac{\ln(t_{11}x + t_{12}y)}{\ln \rho_1}, \tag{21}$$

where $t_{11}, t_{12}, t_{21}, t_{22}$ are the elements of the matrix that brings the matrix with elements $A_{10}, A_{01}, B_{10}, B_{01}$ to diagonal form; ρ_1, ρ_2 are the diagonal elements. If, for simplicity, we assume that w_1 does not depend on F_1, then we find

$$\omega(x, y) = \frac{\omega_{10}x + \omega_{01}y}{1 + \partial w_2(H_1, F_1)/\partial F_1}, \qquad \sigma(x, y) = \frac{\sigma_{10}x + \sigma_{01}y}{1 + \partial w_2(H_1, F_1)/\partial F_1}, \tag{22}$$

where $\omega_{10}, \omega_{01}, \sigma_{10}, \sigma_{01}$ are constant coefficients corresponding to the definitions, chosen in (21), of the logarithms $\ln \rho_1$ and $\ln \rho_2$. From this one can see the type of singularity of the functions $\omega(x, y)$ and $\sigma(x, y)$ at the origin: a periodic function of the logarithm of a linear function. In the case $w_2(H_1, F_1) = w_2(H_1)$, the functions $\omega(x, y)$ and $\sigma(x, y)$ are linear.

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Note: Figure translations are in progress. See original paper for figures.

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