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Abstract

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THE METHOD OF SEPARATION OF VARIABLES IN THE PROBLEM OF SCATTERING BY A BODY OF ARBITRARY SHAPE

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The method of separation of variables is usually applied in mathematical physics only in those cases when the domain under consideration is bounded by coordinate surfaces. For equations of hyperbolic type, a generalization of this method to the mixed problem with an arbitrary initial surface was given in papers ([1,2]). In the present note we investigate the possibility of applying the method of separation of variables to the stationary problem of scattering by a body of arbitrary shape, for any ratio between the wavelength and the dimensions of the body.

1. Let the scattering body be bounded by a closed piecewise differentiable surface S.

Consider outside S a solution of the equation

$$\Delta\psi + k^2\psi = 0, \tag{1}$$

satisfying conditions which will be stated later. On the sphere $r = R > h = \max_S\{r\}$, ψ is expanded in the series

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} X_{lm}(r) Y_{lm}(\vartheta, \varphi) \tag{2}$$

in spherical functions ([3])

$$Y_{lm}(\vartheta, \varphi) = \sqrt{\frac{(l-|m|)!\,(2l+1)}{(l+|m|)!\,4\pi}}\, P_l^{|m|}(\cos\vartheta)\, e^{im\varphi}, \tag{3}$$

whose coefficients

$$X_{lm}(r) = c_{lm}^{+} X_l^{+}(r) - c_{lm}^{-} X_l^{-}(r) \tag{4}$$

are solutions of the radial equations

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dX_l}{dr}\right)+\left(k^2-\frac{l(l+1)}{r^2}\right)X_l=0. \tag{5}$$

Subjecting two linearly independent solutions of equation (5) to the conditions

$$X_l^{\mp}(r)=X_l^{\pm *}(r), \tag{6}$$

$$X_l^{+}X_l^{-\prime}-X_l^{+\prime}X_l^{-}=\frac{2}{ikr^2}, \tag{7}$$

we have

$$X_l^{\pm}(r)=\sqrt{\frac{\pi}{2kr}}\,H_{l+1/2}^{(1)(2)}(kr). \tag{8}$$

2. Applying in the domain between the surface S and the sphere $r=R$ the Gauss-Ostrogradsky formula to the expression $\psi\Delta\varphi-\varphi\Delta\psi$, where

$$\varphi=\varphi_{lm}^{\pm}=X_l^{\pm *}(r)\,Y_{lm}^{*}(\vartheta,\varphi), \tag{9}$$

we obtain the following expressions for the amplitudes of converging and diverging ···

waves in terms of the values of the function ψ and its normal derivative on the surface S:

$$c_{lm}^{\pm}=\frac{ik}{2}\iint\limits_S\left(\psi\frac{\partial\varphi_{lm}^{\pm}}{\partial n}-\varphi_{lm}^{\pm}\frac{\partial\psi}{\partial n}\right)dS. \tag{10}$$

The scattering problem can now be posed as follows: given a sequence of amplitudes of converging waves c_{lm}^{-}, $l=0,1,2,\dots,\ m=-l,\dots,l$, and a linear relation between ψ and $\partial\psi/\partial n$ on S, find the sequence of amplitudes of diverging waves c_{lm}^{+}. Let, for simplicity, $\psi|_S=0$, and denote

$$\left.\frac{\partial\psi}{\partial n}\right|_S=q.$$

Then the scattering problem is formulated as the problem of determining the functionals of q

$$c_{lm}^{+} = \frac{k}{2i} \iint\limits_{S} \varphi_{lm}^{+} q \, dS \tag{11}$$

from the given sequence of other functionals

$$c_{lm}^{-} = \frac{k}{2i} \iint\limits_{S} \varphi_{lm}^{-} q \, dS \tag{12}$$

of the same unknown function q. In particular, for a plane incident wave,

$$c_{lm}^{-} = i^{l} \sqrt{\pi(2l+1)}. \tag{13}$$

An analogous formulation of the problem in the case of scattering by a periodic surface is given in ([4]). To construct ψ in the region $r < h$, one may use, for example, the radial analogue of the Laplace transform ([5]) over the limits from S to ∞ ([4]).

3. Expanding q in a series with respect to some complete system of functions reduces (12) to an infinite system of algebraic equations for the coefficients of this expansion. Subsequent substitution of the series into (11) gives the amplitudes of the scattered waves. Here the essential circumstance is that, effectively, only waves with $l \lesssim kr_0$ participate in scattering by a body of characteristic radius r_0 ([6]). Therefore, in practice, in order to determine c_{lm}^{+} one has to deal with a finite system of linear algebraic equations, the number of which is the smaller the smaller kr_0 is. In the case of axial symmetry this number exceeds kr_0 by only 3–4 units. The rapid practical convergence according to this rule has been verified by concrete calculations in problems of scattering of a plane wave by an ellipsoid of revolution and by a sinusoidal surface. As k increases, the efficiency of the method decreases.

4. The method described extends without special difficulty to the scattering problem with the Schrödinger equation for an arbitrary potential $U(x, y, z)$, whose noncentral part is finite. In this case the functions $X_l^{\pm}(r)$, satisfying conditions (6)–(7), will be solutions of an equation more general than (5), and the expressions for c_{lm}^{+} will contain volume integrals over that part of space in which the potential is noncentral.

5. The formulation of the scattering problem proposed in this note proves, in some respects, preferable to the usual formulation based on the radiation principle ([7]).

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CITED LITERATURE

1. R. G. Barantsev, *Vestn. LGU*, **13**, No. 19 (1958).

2. R. G. Barantsev, *Nauchn. dokl. vyssh. shkoly, fiz.-matem. nauki*, No. 2 (1958).

3. D. I. Blokhintsev, *Fundamentals of Quantum Mechanics*, Moscow, 1961.

4. R. G. Barantsev, *Akust. zhurn.*, **7**, No. 2 (1961).

5. V. A. Ditkin, *DAN*, **116**, No. 1 (1957).

6. H. Massey, C. Mohr, *Proc. Roy. Soc.*, A **141**, No. 844 (1933).

7. N. Mott, H. Massey, *Theory of Atomic Collisions*, Moscow, 1951.

Note: Figure translations are in progress. See original paper for figures.