S. N. CHERNIKOV

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Abstract

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S. N. CHERNIKOV

CONDITIONS FOR THE LOCAL FINITENESS OF SINGLE-LAYER p-GROUPS

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A p-group that contains no elements with orders exceeding the number p will be called **single-layer** in the present note.

With the aid of a theorem of A. I. Kostrikin ([1]) concerning single-layer p-groups, below we obtain necessary and sufficient conditions for the local finiteness of such groups (see Corollary 2 to Theorem 1 and Corollary 1 to Theorem 2). From these conditions and P. S. Novikov' s theorem ([2]) on the existence of infinite periodic groups with a finite number of generating elements, in particular single-layer p-groups of this kind (for $p > 72$), there follows the existence of single-layer simple p-groups (see Corollary 2 to Theorem 2). Thus the question posed by A. G. Kurosh in the article ([3]) on the existence of simple p-groups is answered affirmatively.

Since infinite simple groups obviously have no proper subgroups of finite index, and since no single-layer p-group distinct from the identity can be complete (in the sense of the definition of completeness proposed by the author ([4])), this at the same time gives a positive answer to the question, posed in the article ([5]), on the existence of incomplete groups having no proper subgroups of finite index (such groups we shall call quasi-complete).

1. Lemma 1. *The complete direct product*

$$\mathfrak{P} = \prod_{\alpha \in M} \mathfrak{P}_\alpha$$

of finite single-layer p-groups $\mathfrak{P}_\alpha$ with one and the same number p is a locally finite group.

Proof. Let $P_1, \dots, P_k$ be some finite set of elements of the product $\mathfrak{P}$; let $\mathfrak{P}^*$ be the group generated by them, and let $\mathfrak{P}'_\alpha$ be the subgroup generated by the components of these elements in the set $\mathfrak{P}_\alpha$. By A. I. Kostrikin' s theorem there will be only a finite number of nonisomorphic groups among the subgroups $\mathfrak{P}'_\alpha$. Therefore the product

$$\mathfrak{P}' = \prod_{\alpha \in M} \mathfrak{P}'_\alpha$$

can be represented as a direct product of a finite number of groups, each of which is a complete direct (or perhaps simply direct) product of mutually isomorphic finite groups.

Since each of these groups is locally finite ([6]), their product $\mathfrak{P}'$ is also locally finite. But then the group $\mathfrak{P}^* \subset \mathfrak{P}'$ must be finite (in view of the fact that it has the finite set of generators $P_1, \dots, P_k$). In view of the arbitrariness of the choice of the finite set of elements $P_1, \dots, P_k$ from $\mathfrak{P}$, the local finiteness of the group $\mathfrak{P}$ follows. The lemma is proved.

Corollary. *The complete direct product of locally finite single-layer p-groups is locally finite.*

Remark to Lemma 1. A. I. Kostrikin' s theorem can be strengthened as follows.

If π is a given finite set of prime numbers and h and k are given natural numbers, then among finite π-groups with k generating elements

⋯which have normal series of length not exceeding h, with single-layer factors, there exists a group of some maximum order depending only on π, h, and k.

From the proof of the lemma it is clear that it is valid in the case of the complete direct product of the finite groups considered here, with fixed π and h.

Theorem 1. *Every single-layer p-group $\mathfrak{P}$ is an extension of a single-layer p-group, which has no locally finite factor groups distinct from the identity, by means of a locally finite single-layer p-group.*

Proof. Indeed, let $\mathfrak{R}$ be the intersection of the normal divisors of the group $\mathfrak{P}$ that determine its locally finite factor groups. Consider the factor group $\mathfrak{P}/\mathfrak{R}$. Since the intersection of all normal divisors of the group $\mathfrak{P} = \mathfrak{P}/\mathfrak{R}$ that determine its locally finite factor groups is obviously equal to the identity subgroup $\mathfrak{R}/\mathfrak{R}$, it can be embedded isomorphically in the complete direct product of such factor groups. If

$$[\overline{\mathfrak{M}}_\alpha] = (\dots, \overline{\mathfrak{M}}_\alpha, \dots)$$

is the system of these normal divisors of the group $\mathfrak{P}$, then such a mapping is determined by the correspondence

$$\overline{P} \to [\overline{\mathfrak{M}}_\alpha P],$$

where $\overline{P}$ is an arbitrary element of the group $\mathfrak{P}$.

Since, by Corollary 1 of the lemma, all subgroups of the product under consideration are locally finite, it follows that the factor group $\mathfrak{P} = \mathfrak{P}/\mathfrak{R}$ is locally finite.

If the group $\mathfrak{R}$ has a locally finite factor group distinct from the identity, then the intersection $\mathfrak{R}^*$ of all its normal divisors that determine its locally finite factor groups is distinct from $\mathfrak{R}$, and, by the preceding, the factor group $\mathfrak{R}/\mathfrak{R}^*$ is locally finite. Since the subgroup $\mathfrak{R}^*$ is, obviously, a characteristic subgroup of the normal divisor $\mathfrak{R}$ of the group $\mathfrak{P}$, it is invariant in $\mathfrak{P}$. But then from the local finiteness of the groups $\mathfrak{R}/\mathfrak{R}^*$ and $\mathfrak{P}/\mathfrak{R}$ it follows that the factor group $\mathfrak{P}/\mathfrak{R}^*$ is locally finite (7). Since $\mathfrak{R}^* \neq \mathfrak{R}$, this contradicts the choice of the subgroup $\mathfrak{N}$. Consequently, the group $\mathfrak{N}$ has no locally finite factor groups distinct from the identity. The theorem is proved.

Corollary 1. *Every single-layer p-group with a finite set of generating elements is a finite extension of a quasicyclic single-layer p-group.*

Corollary 2. *For the local finiteness of a single-layer p-group it is necessary and sufficient that every nonidentity subgroup of it (here one may restrict oneself only to normal divisors) have a locally finite factor group distinct from the identity.*

The sufficiency of this condition follows from Theorem 1; the necessity is obvious.

2. Lemma 2. *If a nonidentity group $\mathfrak{G}$ with a finite number of generating elements possesses a normal system with locally finite factors, then it has a proper subgroup of finite index.*

Indeed, each of the generating elements of the group $\mathfrak{G}$ determines a unique jump of the normal system under consideration. Considering it oriented (by inclusion) from left to right (i.e., of any two of its terms, the "larger" is considered to be situated to the right of the "smaller"), we choose from these jumps the rightmost one. If $\mathfrak{N}_\alpha$ and $\mathfrak{N}_{\alpha+1}$ ($\mathfrak{N}_\alpha \subset \mathfrak{N}_{\alpha+1}$) are the members of the system determining it, then, obviously, $\mathfrak{N}_{\alpha+1} = \mathfrak{G}$. But then $\mathfrak{N}_\alpha$ is a proper subgroup of finite index of the group $\mathfrak{G}$.

Theorem 2. *Every single-layer p-group possessing a normal system with locally finite factors is locally finite.*

Proof. Let $\mathfrak{A}$ be an arbitrary subgroup of some single-layer p-group $\mathfrak{P}$ possessing such a normal system, generated by a finite set of elements from $\mathfrak{P}$. Intersections of subgroups-

groups $\mathfrak{A}$ with the members of the normal system under consideration constitute a normal system with locally finite factors for the subgroup $\mathfrak{A}$ ([8]).

By virtue of Corollary 1 of Theorem 1, the group $\mathfrak{A}$ is a finite extension of some quasi-complete subgroup $\mathfrak{A}^*$. As is known, every normal divisor of finite index of a group with a finite set of generators has a finite set of generators (see, for example, ([9])). Therefore the subgroup $\mathfrak{A}^*$ has a finite set of generators. On the other hand, it has a normal system with locally finite factors (since the group $\mathfrak{A}$ has such a system). But then, in view of Lemma 2, the group $\mathfrak{A}^*$ either has a proper subgroup of finite index, or coincides with the identity subgroup of the group $\mathfrak{A}$. Since $\mathfrak{A}^*$ is a quasi-complete group, it follows that $\mathfrak{A}^*$ is the identity group. At the same time it has been proved that the group $\mathfrak{A}$ is finite. Since $\mathfrak{A}$ is an arbitrary subgroup of the group $\mathfrak{P}$, generated by a finite set of its elements, the local finiteness of the group $\mathfrak{P}$ is proved. Thus the theorem is proved.

Corollary 1. *In order that a monostratic p-group be locally finite, it is necessary and sufficient that it possess a normal system with abelian factors.*

The sufficiency of this condition follows from Theorem 2, and the necessity from the theorem on the nonsimplicity of locally finite p-groups ([10]).

Corollary 2. *There exist monostratic simple p-groups.*

Indeed, if all monostratic p-groups were nonsimple, then every monostratic p-group would possess a normal system with cyclic factors. But then it would follow from Theorem 2 that every monostratic p-group is locally finite. This, however, contradicts the main result of the paper by P. S. Novikov ([2]).

Corollary 3. *There exist simple quasi-complete monostratic p-groups.*

This result is contained in Corollary 2, since every infinite simple group is, obviously, quasi-complete.

3. Let now $\mathfrak{P}$ be an infinite monostratic p-group with a finite set of generators. The existence of such groups was established by P. S. Novikov ([2]). In view of Corollary 1 of the theorem, the group $\mathfrak{P}$ is a finite extension of a quasi-complete monostratic p-subgroup $\mathfrak{R}$ distinct from the identity. Thus there exists an infinite quasi-complete p-group whose element orders are bounded in the aggregate (by the number p). Such a group cannot be complete. Indeed, since every element of a complete p-group can, for any natural number n, be represented as a product of p^n-th powers of some of its elements ([4]), a complete p-group must contain elements of arbitrarily large orders. Consequently, among monostratic p-groups there exist quasi-complete but not complete groups.

4. We shall call a group $\mathfrak{G}$ a K-**group** if the finite subgroups of all factor groups of all its subgroups belong to some class K of finite groups. We shall call a periodic group **strongly periodic** if it is a K-group for a class K satisfying the condition (we shall call it condition L): every periodic subgroup of any complete direct product of groups of the class K is locally finite. Such a class is, for example, the class of finite monostratic p-groups with fixed number p (see Lemma 1)*.

Obviously, the following assertion is true: every periodic group of a complete direct product of strongly periodic locally finite K-groups with one and the same K is locally finite. If in the proof of Theorem 1 the reference to the corollary of Lemma 1 is replaced by a reference to this assertion, then from it there follows:

* In the note to Lemma 1 a broader class of finite groups satisfying condition L is indicated.

Theorem 1*. Every strongly periodic group contains a normal divisor having no locally finite factor groups different from the identity, with respect to which it defines a locally finite factor group.

For strongly periodic groups, this theorem implies propositions analogous to Corollaries 1 and 2 of Theorem 1. With the aid of the first of them one proves (analogously to Theorem 2).

Theorem 2*. Every strongly periodic group possessing a normal system with locally finite factors is locally finite.

Let K be a fixed class of finite groups. If it is known that every periodic K-group possessing a normal system with finite factors is locally finite, then, obviously, the class K satisfies condition L. But then every periodic K-group will be strongly periodic, and therefore from Theorem 2^* it follows

Theorem 3. If every strongly periodic K-group (where the class K is fixed) possessing a normal system with finite factors is locally finite, then every strongly periodic K-group possessing a normal system with locally finite factors is also locally finite.

Finally, let us note that from the preceding arguments it follows that the question of the local finiteness of an arbitrary periodic K-group possessing a normal system with locally finite factors is equivalent to the question of the local finiteness of periodic subgroups of arbitrary complete direct products of groups of the class K.

Sverdlovsk Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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