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Abstract

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ON THE TRANSFORMATION OPERATOR FOR STURM-LIOUVILLE DIFFERENTIAL EQUATIONS IN THE NON-SELF-ADJOINT CASE

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1. In this work we shall deal with the Sturm-Liouville differential equation on the half-line. We shall denote the solution of the equation

$$L[u] + \lambda^2 u \equiv u'' - q(x)u + \lambda^2 u = 0 \tag{1}$$

under the boundary conditions

$$u'(0) = h, \qquad u(0) = 1 \tag{1a}$$

by $\omega_h(x,\lambda)$. In particular, the number h may tend to infinity, and in this case the corresponding condition at zero takes the form $u(0) = 0$, $u'(0) = 1$. Denote by $V_{[LL_1hh_1]}$ the transformation operator which takes a solution of problem (1), (1a) into the solution $\omega_{h_1}(x,\lambda)$ of the equation $L_1[y] + \lambda^2 y \equiv y'' - q_1(x)y + \lambda^2 y = 0$. As is known, such an operator exists and is represented in the form *

$$V[f] = f(x) + \int_0^x H(x,t)f(t)\,dt.$$

For the application of the transformation operator in some functional space on the half-line, it is necessary to know more precisely its properties, which are connected with the behavior of the kernel at infinity. Such an investigation was carried out by V. A. Marchenko (^{4}) under the assumption that

$$\int_0^\infty (1+x^2)|q(x)|\,dx < \infty \tag{*}$$

in the case of the self-adjoint problem.

The present work is devoted to the non-self-adjoint case, i.e., neither the potentials nor the initial conditions will be assumed real. In addition, condition (*) is replaced by the less restrictive condition

$$\int_0^\infty x|q(x)|\,dx < \infty. \tag{2}$$

The case is investigated in which h and h_1 may tend to infinity only simultaneously. The case when $h = \infty$, $h_1 \neq \infty$ (or $h_1 = \infty$, $h \neq \infty$) leads to integral operators of the first kind and is not considered here. However, the study of this case can be carried out by the method indicated in the paper.

Under condition (2) there exists a principal solution of equation (1), i.e., a solution satisfying the relation $\lim_{x\to\infty} |y(x,\lambda) - e^{i\lambda x}| = 0$ $(-\infty < \lambda < \infty)$, and it is representable in the form (^{5})

$$y(x,\lambda) = e^{i\lambda x} + \int_0^\infty K(x,t)e^{i\lambda t}\,dt. \tag{3}$$

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* These operators were introduced by B. M. Levitan (^{1}) and A. Ya. Povzner (^{2}) in the study of the generalized shift operator. A systematic exposition of the theory of these operators and their application was given by V. A. Marchenko (^{3}).

The representation (3) for the principal solution and the properties of the kernel $K(x,t)$ (6) played the main role in the proof of the following theorems.

2. The squares of the roots of the equation $y'(0,\lambda) - hy(0,\lambda) = 0$ with positive imaginary part form the discrete spectrum of the operator $-L[u]$ under the boundary condition $u'(0) - hu(0) = 0$. Let us renumber the eigenvalues in such a way that

   $$\operatorname{Im}\lambda_1 \geq \operatorname{Im}\lambda_2 \geq \cdots \geq \operatorname{Im}\lambda_l \geq \cdots > 0,$$

   and, if $\operatorname{Im}\lambda_{j-1} = \operatorname{Im}\lambda_j$, then the multiplicity k_j of the root λ_j is not less than the multiplicity k_{j-1} of the root λ_{j-1}.

Theorem 1. Let $q(x)$ satisfy condition (2), and let

$$\operatorname{Im}\lambda_{l+1} < a < \operatorname{Im}\lambda_l.$$

Then, on the class of functions $f(x)$ satisfying the condition

$$\int_0^\infty |f(x)|e^{-\operatorname{Im}\lambda_l x}x^{k_l-1}\,dx < \infty, \tag{4}$$

one can define an operator transforming the solution of problem (1), (1a) for $|\operatorname{Im}\lambda| < \operatorname{Im}\lambda_l$ into the solution of the equation $z'' + \lambda^2 z = 0$ under the boundary condition $z'(0) = h,\ z(0) = 1$. This operator is represented in the form*

$$V^{+}_{[LD^2hh_1]}[f] = f(x) + e^{ax}\int_0^x K_a(x,t)f(t)\,dt + \int_0^\infty \sum_{j=1}^{l} S_j(x,t)f(t)\,dt,$$

where

$$\sup_{0\le x<\infty} \int_0^x K_a(x,t)\,dt < \infty,$$

$$\sup_{0\le x<\infty} \frac{1}{1+\varphi(x)} \int_0^\infty |S_j(x,t)|(1+\varphi(t))\,dt < \infty \qquad (j = 1, \dots, l)$$

for any weight** $\varphi(x)$ satisfying condition (4).

If, however, one takes $a > \operatorname{Im}\lambda_1$, then the operator

$$V_{[LD^2hh_1]}[f] = f(x) + e^{ax}\int_0^x K_a(x,t)f(t)\,dt$$

is a transformation operator (i.e., transforms in the indicated manner the solution for all λ). The theorem also holds for $h = h_1 = \infty$.

Let us outline the proof of Theorem 1. Consider the generalized Fourier transform (5)

$$\psi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\,\frac{i\lambda - h_1}{y'(0,\lambda) - hy(0,\lambda)}\,y(x,\lambda)\,dx, \tag{5}$$

in which $\operatorname{Im}\lambda = a,\ \operatorname{Im}\lambda_l > a > \operatorname{Im}\lambda_{l+1}$. Using representation (3) for $y(x,\lambda)$, we obtain the ordinary Fourier transform of $\psi(\lambda)$. On the other hand, if in (5) one substitutes for $f(x)$ the solutions $\omega_h(x,\mu)$, $|\operatorname{Im}\mu| \le a$, then the integral is evaluated directly. Taking then its Fourier transform and comparing with that obtained by the preceding method, one can obtain the assertion of Theorem 1. In doing so, the properties of the kernel of the operator $V^{+}_{[LD^2hh_1]}$ can be clarified by relying on the properties of the kernel $K(x,t)$ in representation (3) for the principal solution.

Corollary. If $y'(0,\lambda) - hy(0,\lambda)$ is nonzero for real λ, then there exists only a finite number of points of the discrete spectrum, and Theorem 1 is valid for $a = 0$. In this case l is the number of eigenvalues of the boundary-value problem.

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* By D^2 is denoted the operator d^2/dx^2.

** By a weight $\varphi(x)$ we mean a nonnegative nondecreasing function satisfying the condition

$$\varphi(x+y) \le \varphi(x)\cdot\varphi(y)$$

for $x > x_0,\ y > x_0$.

3. For what follows it is essential that, for $t > x$, the kernels $S_j(x,t)$ occurring in the formulation of Theorem 1 have the form

$$S_j(x,t) = \sum_{k=1}^{k_j} \frac{i}{(k-1)!}\, c_{kj}\, [e^{-ix\lambda} y(t,\lambda)]_\lambda^{(k-1)}\Big|_{\lambda=\lambda_j},$$

where the coefficients c_{kj} are computed from the eigenfunctions and associated functions of the boundary-value problem. Moreover, it can be shown that the eigenfunctions and associated functions of the boundary-value problem

$$-L[u] = \lambda^2 u$$

with the boundary condition

$$u'(0) - hu(0) = 0,$$

corresponding to the eigenvalues λ_j^2, are biorthogonal on the positive half-axis to the solution $\omega_h(x,\mu)$, if and only if

$$|\operatorname{Im}\mu| < \operatorname{Im}\lambda_j.$$

With the aid of these remarks, the following assertion can be obtained from Theorem 1.

Theorem 2. *Let $q(x)$ satisfy condition (2). Then the transformation operator $V_{[LD^2hh_1]}$ can, on the space of functions $f(x)$ satisfying condition (4), be represented in the form*

$$V_{[LD^2hh_1]} = V^-_{[LD^2hh_1]} + V^+_{[LD^2hh_1]},$$

where

$$V^-_{[LD^2hh_1]}[f] = -\int_0^\infty \sum_{j=1}^{l} \sum_{k=1}^{k_j} \frac{i}{(k-1)!}\, c_{kj}\, [e^{-ix\lambda} y(t,\lambda)]_\lambda^{(k-1)}\Big|_{\lambda=\lambda_j} f(t)\, dt,$$

and the operator $V^+_{[LD^2hh_1]}$ is the same as in Theorem 1.

The theorem is valid in full also for the operator $V_{[LD^2\infty\infty]}$.

Corollary. *If $y'(0,\lambda) - hy(0,\lambda)$ is different from zero for real λ, then the operator $V^+_{[LD^2hh_1]}$ can be constructed as indicated in the corollary to Theorem 1. In the kernel of the operator $V^-_{[LD^2hh_1]}$ in this case, the summation is carried out over all eigenvalues λ_j^2!*

In the self-adjoint case, the corollary of Theorem 2 in its most essential part coincides with Theorem 4.3.1 of V. A. Marchenko (4).

4. The study of the transformation operators $V_{[D^2Lhh]}$ and $V_{[D^2L\infty\infty]}$ turns out to be simpler than that of the operator $V_{[LD^2hh_1]}$. However, the basic idea in the proof of the corresponding theorem is the same as in the proof of Theorem 1. It is only necessary to use the inversion formula (5) for transformation (5). As a result one obtains:

Theorem 3. *Under condition (2) on the potential* $q(x)$*, the Volterra transformation operators* $V_{[D^2L\infty\infty]}$ *and* $V_{[D^2Lh_1h]}$*, where* $\operatorname{Re} h_1 > 0$**, are bounded in the space* $M_1[0,\infty)$ *of functions bounded on the half-axis.*

In conclusion, we note that the investigation of more general transformation operators $V_{[L_1L_2h_1h_2]}$ (h_1 and h_2 tend to infinity only simultaneously) on the positive half-axis reduces to those considered here, since any transformation operator can be represented in the form

$$V_{[L_1L_2\infty\infty]} = V_{[L_1D^2\infty\infty]}V_{[D^2L_2\infty\infty]},$$

$$V_{[L_1L_2h_1h_2]} = V_{[L_1D^2h_1h]}V_{[D^2L_2hh_2]} \qquad (h_1 \neq \infty,\ h_2 \neq \infty),$$

where $\operatorname{Re} h > 0$.

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* If $h_1 = 0$, the assertion of the theorem ceases to be true. In this case the construction of the operator is mo

Note: Figure translations are in progress. See original paper for figures.