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Abstract

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I. I. SHMULEV

BOUNDED SOLUTIONS OF BOUNDARY-VALUE PROBLEMS WITHOUT INITIAL CONDITIONS FOR PARABOLIC EQUATIONS AND INVERSE BOUNDARY-VALUE PROBLEMS

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Bounded classical solutions of the Dirichlet and Neumann problems without initial conditions for the equation

$$u_t = \Delta u + qu + F(x,t) \qquad (q = \text{const})$$

were studied in ([1]). In the present note, for the study of these problems both for linear and for nonlinear parabolic equations, a simple method is proposed, borrowed from the theory of elliptic equations and originating in the works ([2–4]).

Let D be a bounded m-dimensional domain of the space of variables $(x_1, \dots, x_m) = x$, and let Γ be the boundary of D. Denote by $Q = D \times (-\infty, +\infty)$ the straight cylinder of $m+1$ dimensions, and by S the lateral surface of Q. The part of Q enclosed between the planes $t = t_1$ and $t = t_2$, where t_1 and t_2 are arbitrary numbers from $(-\infty, +\infty)$ and $t_2 > t_1$, will be denoted by $Q_{[t_1,t_2]}$, and by $S_{[t_1,t_2]}$ the lateral surface of $Q_{[t_1,t_2]}$. Finally, denote by L the elliptic operator

$$L = \sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t),$$

where

$$\sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \ge \alpha \sum_{i=1}^{m} \xi_i^2, \quad a_{ij}(x,t) = a_{ji}(x,t) \quad \text{and} \quad -c(x,t) \ge c_0,$$

for $(x,t) \in \overline{Q}$; α and c_0 are positive numbers.

Below, by a solution of one or another problem we shall mean a classical solution of the problem under consideration.

Consider in Q the Dirichlet problem without initial conditions:

$$u_t = Lu + f(x,t), \tag{1}$$

$$u\big|_S = \varphi(x,t). \tag{2}$$

Theorem 1. Let the coefficients of L and the bounded function $f(x,t)$ be such that, in any $Q_{[t_1,t_2]}$, the boundary-value problem

$$u_t = Lu + f(x,t), \qquad u\big|_{S_{[t_1,t_2]}} = \Phi(x,t), \qquad u(x,t_1) = \Psi(x)$$

has a solution, whatever the continuous functions $\Phi(x,t)$, $\Psi(x)$ may be ($\Psi|_\Gamma = \Phi$). Then, if $\varphi(x,t)$ is continuous and bounded, problem (1), (2) has in Q a unique bounded solution.

We indicate a brief scheme of the proof. Let $v(x,t)$ be the solution in $Q^- = D \times (-\infty, 0]$ of the boundary-value problem

$$v_t = Lv + \lambda v + e^{\lambda t} f(x,t), \tag{3}$$

$$v\big|_{S^-} = e^{\lambda t}\varphi(x,t), \tag{4}$$

admitting in $\overline{Q^-}$ the estimate

$$|v(x,t)| = O\left(e^{\lambda t}\right). \tag{5}$$

The number $\lambda > 0$ is chosen according to the condition $c_0 - \lambda > 0$. We shall prove the existence of $v(x,t)$.

Construct in Q^- a sequence $\{v_n(x,t)\}$ as follows. Let $\{Q_n^-\}$ be a sequence of nested cylinders:

$$Q_n^- = D \times [t_n, 0], \qquad t_n \to -\infty.$$

Define $v_n(x,t)$ in Q_n^- as the solution of the boundary-value problem

$$v_{nt} = Lv_n + \lambda v_n + e^{\lambda t} f(x,t), \tag{6}$$

$$v_n\big|_{S_{[t_n,0]}} = e^{\lambda t}\varphi_n(x,t), \tag{7}$$

$$v_n(x, t_n) = 0, \tag{8}$$

where $\varphi_n(x,t)$ is defined by the equalities

$$\varphi_n(x,t) = \begin{cases} \varphi(x,t), & t_n + \delta \le t \le 0, \\ \varphi(x,t)\cos\left(\dfrac{t - t_n - \delta}{\delta}\right)\dfrac{\pi}{2}, & t_n \le t \le t_n + \delta, \end{cases}$$

in which the number $\delta > 0$ is sufficiently small. In the domain $\overline{Q^-} - \overline{Q_n^-}$ set $v_n(x,t) = 0$.

By the hypothesis of the theorem, problem (6)–(8) has a solution in Q_n^-, for which, as is not difficult to show, estimate (5) is valid. With the aid of this estimate one then establishes the uniform convergence in $\overline{Q^-}$ of the sequence $\{v_n(x,t)\}$ to some function $v(x,t)$, which, as is shown, will be a solution of equation (3) in every bounded part of Q^-.

It is immediately clear that the function $u^- = ve^{-\lambda t}$ will be a bounded solution in Q^- of the boundary-value problem:

$$u_t^- = Lu + f(x,t), \qquad u^-\big|_{S^-} = \varphi(x,t),$$

where S^- is the lateral surface of Q^-.

Define $u^+(x,t)$ as the solution in $Q^+ = D \times [0, +\infty)$ of the boundary-value problem:

$$u_t^+ = Lu^+ + f(x,t), \tag{9}$$

$$u^+\big|_{S^+} = \varphi(x,t), \tag{10}$$

$$u^+(x,0) = u^-(x,0), \tag{11}$$

where S^+ is the lateral surface of Q^+. The function

$$u(x,t) = \begin{cases} u^+(x,t), & 0 \le t < +\infty, \\ u^-(x,t), & -\infty < t \le 0, \end{cases}$$

is a bounded solution of problem (1), (2).

We shall prove the uniqueness of the bounded solution of problem (1), (2). Let u_1 and u_2 be two bounded solutions of problem (1), (2), and let $u = u_2 - u_1$.

Denote by u^- the part of u defined in Q^-. The function $v = e^{\lambda t}u^-$, where $0 < \lambda < c_0$, is a solution in Q^- of the boundary-value problem

$$v_t = Lv + \lambda v, \qquad v\big|_{S^-} = 0.$$

Whatever $T \in (-\infty, 0)$ may be, in $Q^-_{[T,0]}$ the estimate $|v(x,t)| = O(e^{\lambda T})$ holds, following from the maximum principle for parabolic equations. The indicated estimate makes it possible to conclude that in Q^- the function $v = 0$, i.e. $u^- = 0$. But then, on the basis of the uniqueness theorem for the problem

$$u_t = Lu, \tag{12}$$

(9)–(11), the part u defined in Q^+ is also equal to zero: $u^+ = 0$. Thus in $\overline{Q}$, $u_2 \equiv u_1$. Let us note an important special case of Theorem 1.

Theorem 2. *If the conditions of Theorem 1 are satisfied, and the coefficients L and the functions $f(x,t)$ and $\varphi(x,t)$ are periodic in t with period T, then in Q there exists a unique solution of problem (1), (2) that is periodic in t with period T.*

Let us now consider in Q the Neumann problem without initial conditions:

$$\left(\frac{\partial u}{\partial \gamma} - a(x,t)u\right)\bigg|_S = \varphi(x,t) \qquad (a(x,t) \geqslant a_0 = \text{const} > 0), \tag{13}$$

where γ is the direction of the conormal to S.

Theorem 3. *Let S belong to the class $A^{(2)}$, and let the coefficients L and the continuous and bounded function $a(x,t)$ be such that in any $Q_{[t_1,t_2]}$ the boundary-value problem*

$$u_t = Lu, \qquad (\partial u/\partial\gamma - a(x,t)u)\big|_{S_{[t_1,t_2]}} = \Phi(x,t), \qquad u(x,t_1) = \Psi(x)$$

has a solution, whatever the continuous functions $\Phi(x,t)$ and $\Psi(x)$ may be. Then, if $\varphi(x,t)$ is continuous and bounded, problem (12), (13) has in Q a unique bounded solution.

The proof of this theorem repeats the main points of the proof of Theorem 1; however, in addition to the maximum principle for equation (12), Theorem 1 of paper ([5]) is also used.

Theorem 4. *If the conditions of Theorem 3 are satisfied, and the coefficients L and the functions $a(x,t)$ and $\varphi(x,t)$ are periodic in t with period T, then in Q there exists a unique solution of boundary-value problem (12), (13) that is periodic in t with period T.*

The method of proof of Theorems 1 and 3, with some modification, may be used in the study of bounded solutions of nonlinear boundary-value problems.

Let us begin with the first boundary-value problem:

$$u_t = \sum_{i,j=1}^{m} a_{ij}(x,t,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t,u)\frac{\partial u}{\partial x_i} + c(x,t,u)u + f(x,t), \quad (14)$$

$$u\big|_S = 0 \qquad (s \in A^{(2,\lambda)}), \quad (15)$$

where

$$\sum_{i,j=1}^{m} a_{ij}(x,t,u)\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{m} \xi_i^2$$

for $(x,t) \in \overline{Q}$ and $u \in (-\infty, +\infty)$; $a_{ij} = a_{ji}$ and $-c(x,t,u) \geqslant c_0$ for the indicated values of x, t, and u; α and c_0 are positive numbers.

Theorem 5. *In the cylinder Q there exists at least one bounded solution of boundary-value problem (14), (15), if the following conditions are satisfied:*

1. *For $(x,t) \in \overline{Q}$, $|u| \leqslant C_0$ $(C_0 = \text{const} > 0)$, both the functions $a_{ij}(x,t,u)$, $b_i(x,t,u)$, $c(x,t,u)$ and their derivatives with respect to x, u up to the fourth order inclusive are continuous, bounded, and satisfy a Hölder condition in x, u, with*
$$\max_{(x,t,u)} |\partial a_{ij}/\partial u| \leqslant \alpha e\sqrt{3}/12\, mC_0\,^*.$$
2. *The function $f(x,t)$ and its derivatives with respect to x up to the fourth order inclusive are continuous and bounded.*

The proof of this theorem uses a priori estimates of solutions and their derivatives for the first boundary-value problem for equation (14), obtained in papers ([6],[7]).

Theorem 6. *In the strip $Q = \{0 < x < l,\ -\infty < t < +\infty\}$ there exists at least one bounded solution of the boundary-value problem*

* This condition can apparently be replaced by the condition that $|\partial a_{ij}/\partial u|$ is simply bounded (see ([8])).

$$u_t = a(x,t,u)_{xx} + b(x,t,u)u_x + c(x,t,u)u + f(x,t), \quad (16)$$

$$u_x(0,t) = \varphi_1(u(0,t),t), \quad (17)$$

$$u_x(l,t) = \varphi_2(u(l,t),t), \tag{18}$$

if the following conditions are satisfied:

1. For $(x,t) \in \overline{Q}$, $|u| \leq C_0$ ($C_0 = \text{const}$), the functions $a(x,t,u)$, $b(x,t,u)$, $c(x,t,u)$, $f(x,t)$ and their derivatives with respect to x, u up to and including the fourth order are continuous and bounded, and the derivatives of second order with respect to x, u have bounded first-order derivatives with respect to t.
2. For $(x,t) \in \overline{Q}$, $u \in (-\infty,+\infty)$, the inequalities $a(x,t,u) \geq \alpha$, $-c(x,t,u) \geq c_0$ hold, where α, c_0 are positive numbers.
3. For $t \in (-\infty,+\infty)$, $|u| \leq C_0$, the functions $\varphi_1(u,t)$, $\varphi_2(u,t)$ are continuous and bounded together with their derivatives with respect to u up to and including the second order and with their first-order derivatives with respect to t, and $\varphi_1(0,t) = \varphi_2(0,t) = 0$.
4. For $t \in (-\infty,+\infty)$, $u \in (-\infty,+\infty)$, the inequalities $\varphi_{1u}(u,t) \geq \beta_1$, $-\varphi_{2u}(u,t) \geq \beta_2$ hold, where β_1, β_2 are positive numbers.

The proof of the theorem is based on the results of paper (9).

We indicate the connection between the questions considered and inverse boundary-value problems.

Theorem 7. If the hypotheses of Theorem 1, referred to Q^-, are satisfied, then the inverse boundary-value problem

$$u_t = Lu + f(x,t), \tag{19}$$

$$u|_{S^-} = \varphi(x,t) \qquad (t \in (-\infty,0]), \tag{20}$$

$$u(x,0) = \psi(x) \qquad (\psi|_\Gamma = \varphi) \tag{21}$$

has a bounded solution in Q^- if and only if the continuous function $\psi(x) \equiv u^-(x,0)$, where $u^-(x,t)$ is the solution of problem (19), (20) bounded in Q^-.

Theorem 8. If the hypotheses of Theorem 3, referred to Q^-, are satisfied, then the inverse boundary-value problem

$$u_t = Lu, \tag{22}$$

$$(\partial u/\partial\gamma - a(x,t)u)|_{S^-} = \varphi(x,t) \qquad (t \in (-\infty,0]), \tag{23}$$

$$u(x,0) = \psi(x) \tag{24}$$

has a bounded solution in Q^- if and only if the continuous function $\psi(x) \equiv u^-(x,0)$, where $u^-(x,t)$ is the solution of problem (22), (23) bounded in Q^-.

Since the function $u^-(x,t)$ appearing in each of these theorems is determined uniquely, as was established earlier, it follows from these theorems that the inverse Dirichlet and Neumann boundary-value problems are ill-posed in the class of bounded functions.

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Voronezh Forestry Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.