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Soviet-era science, translated into English

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1962

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**Abstract**

**Full Text**

**I. A. Kipriyanov**

## On a Certain Class of Weighted Embedding Theorems

*(Presented by Academician S. L. Sobolev, 13 VI 1962)*

Let  $R_n$  denote the  $n$ -dimensional Euclidean space of points  $x = (x_1, \dots, x_n)$ . Introduce in  $R_n$  spherical coordinates with origin at a certain point  $x_0 = (x_1^{(0)}, \dots, x_n^{(0)})$ . Then a function  $f(x)$ , defined in the space  $R_n$ , may be regarded as a function of  $x_0, \rho, \omega_n$ , where by  $\rho$  we denote the distance between the points  $x_0$  and  $x$ , and by  $\omega_n$  the collection of angular spherical coordinates. Consider the collection of all functions  $f(x)$  that are finite in the domain  $R_n$  with the point  $x_0$  removed. We call a function  $f(x)$  finite in the domain  $R_n$  with the point  $x_0$  removed if it is continuously differentiable in  $R_n$  and vanishes not only outside some bounded domain (its own for each function  $f(x)$ ) for which the point  $x_0$  is an interior point, but also in a neighborhood (its own for each function  $f(x)$ ) of the point  $x_0$  itself.

On the collection thus introduced define the differential operator

$$D_\rho f = \frac{1}{\rho} \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cos(\rho, x_i) \quad (1)$$

and powers of this operator

$$D_\rho^l f = \frac{\partial}{\rho \partial \rho} \left( \frac{\partial^{l-1} f}{(\rho \partial \rho)^{l-1}} \right) \quad (l = 2, 3, \dots). \quad (2)$$

Define the functional space  $W_{\rho,2}^{(l)}(R_n)$  as the closure of the set of functions  $f(x)$ , finite in the domain  $R_n$  with the point  $x_0$  removed, with respect to the norm

$$\|f\|_{W_{\rho,2}^{(l)}(R_n)}^2 = \int_{R_n} |f|^2 dx + \int_{R_n} \left| \rho^l \frac{\partial^l f}{(\rho \partial \rho)^l} \right|^2 dx, \quad (3)$$

where  $dx = \rho^{n-1} d\rho d\omega_n$ , and  $l$  is a positive integer.

**Theorem 1.** If the function  $f(x)$  belongs to the space  $W_{\rho,2}^{(l)}(R_n)$ , then  $\partial^m f / (\rho \partial \rho)^m$  ( $m < l$ ) is square summable with weight  $\rho^m$  and the inequality

$$\|\rho^m \partial^m f / (\rho \partial \rho)^m\|_{L_2(R_n)} \leq c_1 \|f\|_{W_{\rho,2}^{(l)}(R_n)}. \quad (4)$$

holds.

It follows from this that the norm in  $W_{\rho,2}^{(l)}(R_n)$  defined by the formula

$$\|f\|_{W_{\rho,2}^{(l)}(R_n)}^2 = \int_{R_n} |f|^2 dx + \sum_{k=1}^l \int_{R_n} \left| \rho^k \frac{\partial^k f}{(\rho \partial \rho)^k} \right|^2 dx, \quad (5)$$

will be equivalent to the norm introduced above.

On the set of functions  $f(x)$ , finite in the domain  $R_n$  with the point  $x_0$  removed, introduce for consideration the operator

$$D_\rho^{m+\alpha} f = \frac{1}{\rho^{n-1-\alpha+m}} \frac{\partial}{\partial \rho} \left[ \frac{1}{\Gamma(1-\alpha)} \int_0^\rho (\rho^2 - \tau^2)^{-\alpha} \tau^m \left( \tau^m \frac{\partial^m f(x_0, \tau, \omega_n)}{(\tau \partial \tau)^m} \right) \tau^{n-1} d\tau \right], \quad (6)$$

where  $m$  is a nonnegative integer and  $\alpha$  is a proper fraction ( $0 < \alpha < 1$ ).

The functional space  $W_{\rho,2}^{(l+\alpha)}(R_n)$ , where  $l$  is a nonnegative integer and  $\alpha$  is a proper fraction ( $0 < \alpha < 1$ ), is defined as the closure of the set of functions  $f(x)$ , finite in  $R_n$  with the point  $x_0$  removed, with respect to the norm

$$\|f\|_{W_{\rho,2}^{(l+\alpha)}(R_n)}^2 = \int_{R_n} |f|^2 dx + \int_{R_n} |D_\rho^{l+\alpha} f|^2 dx. \quad (7)$$

Then the following holds.

**Theorem 2.** If  $f \in W_{\rho,2}^{(l+\alpha)}(R_n)$ , then also  $f \in W_{\rho,2}^{(l)}(R_n)$ , and the inequality

$$\|\rho^l \partial^l f / (\rho \partial \rho)^l\|_{L_2(R_n)} \leq c_2 \|f\|_{W_{\rho,2}^{(l+\alpha)}(R_n)} \quad (8)$$

is satisfied.

Therefore, in the space  $W_{\rho,2}^{(l+\alpha)}(R_n)$  one may also introduce a norm by the formula

$$\|f\|_{W_{\rho,2}^{(l+\alpha)}(R_n)}^2 = \|f\|_{W_{\rho,2}^{(l)}(R_n)}^2 + \int_{R_n} |D_\rho^{l+\alpha} f|^2 dx, \quad (9)$$

which will be equivalent to the norm (7).

**Theorem 3.** If  $f \in W_{\rho,2}^{(l)}(R_n)$ , then also  $f \in W_{\rho,2}^{(l_1)}(R_n)$ , where  $l$  and  $l_1$  are positive numbers and  $l_1 < l$ , and the inequality

$$\|D_{\rho}^{l_1} f\|_{L_2(R_n)} \leq c_3 \|f\|_{W_{\rho,2}^{(l)}(R_n)} \quad (10)$$

holds.

It follows from the last theorem that in the space  $W_{\rho,2}^{(l+\alpha)}(R_n)$  the norm defined by the formula

$$\|f\|_{W_{\rho,2}^{(l+\alpha)}(R_n)}^2 = \|f\|_{W_{\rho,2}^{(l)}(R_n)}^2 + \sum_{k=0}^l \int_{R_n} |D_{\rho}^{k+\alpha} f|^2 dx, \quad (11)$$

will be equivalent to the norm (9). In the case  $l = 0$  we understand  $W_{\rho,2}^{(0)}(R_n)$  to mean the space  $L_2(R_n)$ .

Represent  $R_n$  as a topological product  $R_n = R_m \times R_{n-m}$ . In each of the subspaces  $R_m$  and  $R_{n-m}$  introduce spherical coordinates. Then any point  $x \in R_n$  can be represented as a pair  $x = (x^{(m)}, x^{(n-m)})$ , and a function  $f(x)$ , defined on  $R_n$ , can be represented in the form  $f(x) = f(x^{(m)}, x^{(n-m)}) = f(x_0^{(m)}, \rho_1, \omega_m, x_0^{(n-m)}, \rho_2, \omega_{n-m})$ . Here by  $\rho_1$  we denote the distance between the points  $x_0^{(m)}$  and  $x^{(m)}$  in  $R_m$ , by  $\rho_2$  the distance between the points  $x_0^{(n-m)}$  and  $x^{(n-m)}$  in the subspace  $R_{n-m}$ , and by  $\omega_m$  and  $\omega_{n-m}$  the collections of angular spherical coordinates in the corresponding subspaces. The points  $x_0^{(m)}$  and  $x_0^{(n-m)}$  are the origins of spherical coordinates, respectively, in the subspaces  $R_m$  and  $R_{n-m}$ .

Define the functional space  $W_{\rho_1,2}^{(l_1)}(R_n)$ , where  $l_1$  is a nonnegative number, as the closure of the set of functions  $f(x)$ , finite in  $R_n$  with the submanifold  $R_{n-m}$  removed, with respect to the norm

$$\|f\|_{W_{\rho_1,2}^{(l_1)}(R_n)}^2 = \int_{R_n} |f|^2 dx + \int_{R_n} |\widetilde{D}_{\rho_1}^{l_1} f|^2 dx, \quad (12)$$

where, in the case of integer  $l_1$ , by  $\widetilde{D}_{\rho_1}^{l_1} f$  one should understand the operator  $\rho_1^{l_1} \partial^{l_1} f / (\rho_1 \partial \rho_1)^{l_1}$ , while in the case of fractional  $l_1$ , by  $\widetilde{D}_{\rho_1}^{l_1} f$  one should understand the operator  $D_{\rho_1}^{l_1} f$ .

The space  $W_{\rho_2,2}^{(l_2)}(R_n)$  is defined analogously. We shall say that

$$f(x^{(m)}, x^{(n-m)}) \in W_{\rho_1, \rho_2, 2}^{(l_1, l_2)}(R_n),$$

if it belongs to the intersection of the classes  $W_{\rho_1,2}^{(l_1)}(R_n)$  and  $W_{\rho_2,2}^{(l_2)}(R_n)$ . We introduce the norm by the formula

$$\|f\|_{W_{\rho_1,\rho_2,2}^{(l_1,l_2)}(R_n)}^2 = \|f\|_{W_{\rho_1,2}^{(l_1)}(R_n)}^2 + \|f\|_{W_{\rho_2,2}^{(l_2)}(R_n)}^2. \quad (13)$$

**Theorem 4.** Let nonnegative integers  $\alpha$  be given, for which

$$\mu = \mu(\alpha) = 1 - 2\alpha/l_2 - (n - m)/2l_2 > 0. \quad (14)$$

For such  $\alpha$ , define on  $R_m$  functions  $\varphi^{(\alpha)}(x^{(m)}) \in W_{\rho_1,2}^{(\bar{l}_1)}(R_m)$ ,  $\bar{l}_1 = \mu l_1$ . There exists a function  $\bar{f}(x) \in W_{\rho_1,\rho_2,2}^{(l_1,l_2)}(R_n)$  such that

$$\lim_{\rho_2 \rightarrow 0} \left\| \frac{\partial^\alpha \bar{f}(x^{(m)}, x_0^{(n-m)}, \rho_2, \omega_{n-m})}{(\rho_2 \partial \rho_2)^\alpha} - \varphi^{(\alpha)}(x^{(m)}) \right\|_{W_{\rho_1,2}^{(\bar{l}_1)}(R_m)} = 0 \quad (15)$$

for all admissible  $\alpha$ . Moreover, the inequality

$$\|\bar{f}\|_{W_{\rho_1,\rho_2,2}^{(l_1,l_2)}(R_n)} \leq c_4 \sum_{\alpha} \|\varphi^{(\alpha)}(x^{(m)})\|_{W_{\rho_1,2}^{(\bar{l}_1)}(R_m)} \quad (16)$$

holds, where the positive constant  $c_4$  does not depend on the functions  $\varphi^{(\alpha)}$ .

When relation (15) holds, we shall denote the function  $\varphi^{(\alpha)}(x^{(m)})$  by  $\partial^\alpha f / (\rho_2 \partial \rho_2)^\alpha \big|_{\rho_2=0}$ .

The converse theorem is also valid:

**Theorem 5.** Let  $f(x) \in W_{\rho_1,\rho_2,2}^{(l_1,l_2)}(R_n)$ , and suppose that for some nonnegative integers  $\alpha$  inequality (14) is satisfied. Then  $\partial^\alpha f / (\rho_2 \partial \rho_2)^\alpha$ , as a function of  $x^{(m)}$ , with  $\rho_2$  and  $\omega_{n-m}$  fixed in the corresponding manner, belongs to the space  $W_{\rho_1,2}^{(\bar{l}_1)}(R_m)$  with  $\bar{l}_1 = \mu l_1$ , and the inequality

$$\left\| \partial^\alpha f / (\rho_2 \partial \rho_2)^\alpha \big|_{\rho_2=0} \right\|_{W_{\rho_1,2}^{(\bar{l}_1)}(R_m)} \leq c_5 \|f\|_{W_{\rho_1,\rho_2,2}^{(l_1,l_2)}(R_n)} \quad (17)$$

holds, where  $c_5$  does not depend on  $f$ .

Representing  $R_n$  as the topological product  $R_n = R_m \times R_{n-m}$ , we introduce spherical coordinates only in the subspace  $R_m$ . Then a function  $f(x)$  defined on  $R_n$  is written in the form  $f(x) = f(x^{(m)}, x^{(n-m)}) = f(x_0^{(m)}, \rho, \omega_m, x_{m+1}, \dots, x_n)$ , where  $\rho$  is the distance between  $x_0^{(m)}$  and  $x^{(m)}$  in the subspace  $R_m$ , and  $\omega_m$ , as usual, denotes the set of angular spherical coordinates in  $R_m$ . Let  $l_1, l_{m+1}, \dots, l_n$  be nonnegative numbers. We shall say that  $f(x) \in W_{\rho, x_{m+1}, \dots, x_n, 2}^{(l_1, l_{m+1}, \dots, l_n)}(R_n)$ , if  $f(x) \in$

$W_{\rho,2}^{(l)}(R_n)$  and if  $f(x) \in W_{x_i,2}^{(l_i)}(R_n)$  for all  $i = m+1, \dots, n$  (see definition (1)). As usual, we introduce the norm by the formula

$$\|f\|_{W_{\rho, x_{m+1}, \dots, x_n}^{(l, l_{m+1}, \dots, l_n)}(R_n)}^2 = \|f\|_{W_{\rho,2}^{(l)}(R_n)}^2 + \sum_{i=m+1}^n \|f\|_{W_{x_i,2}^{(l_i)}(R_n)}^2. \quad (18)$$

**Theorem 6.** Let nonnegative integers  $r_{m+1}, \dots, r_n$  be given, for which

$$\mu = \mu(r_{m+1}, \dots, r_n) = 1 - \sum_{m+1}^n \frac{r_i}{l_i} - \frac{1}{2} \sum_{m+1}^n \frac{1}{l_i} > 0. \quad (19)$$

For such numbers, define on  $R_m$  functions

$$\varphi^{(\alpha)}(x^{(m)}) = \varphi^{(r_{m+1}, \dots, r_n)}(x^{(m)}) \in W_{\rho,2}^{(\bar{l})}(R_m) \quad \bar{l} = \mu l.$$

There exists a function  $\bar{f} \in W_{\rho, x_{m+1}, \dots, x_n, 2}^{(l, l_{m+1}, \dots, l_n)}(R_n)$  such that

$$\lim_{x^{(n-m)} \rightarrow 0} \|\partial^\alpha \bar{f}(x^{(m)}, x^{(n-m)}) / \partial x_{m+1}^{r_{m+1}} \dots \partial x_n^{r_n} - \varphi^{(\alpha)}(x^{(m)})\|_{W_{\rho,2}^{(\bar{l})}(R_m)} = 0 \quad (20)$$

for all admissible  $r_{m+1}, \dots, r_n$ . Moreover,

$$\|\bar{f}\|_{W_{\rho, x_{m+1}, \dots, x_n, 2}^{(l, l_{m+1}, \dots, l_n)}(R_n)} \leq c_6 \sum_{\alpha} \|\varphi^{(\alpha)}(x^{(m)})\|_{W_{\rho,2}^{(\bar{l})}(R_m)}, \quad (21)$$

where  $c_6$  does not depend on  $\varphi^{(\alpha)}$ .

The converse theorem is also true:

**Theorem 7.** Let  $f \in W_{\rho, x_{m+1}, \dots, x_n, 2}^{(l, l_{m+1}, \dots, l_n)}(R_n)$ , and suppose that for some nonnegative integers  $r_{m+1}, \dots, r_n$  inequality (19) is satisfied. Then the derivatives  $\partial^\alpha f / \partial x_{m+1}^{r_{m+1}} \dots \partial x_n^{r_n}$ , as functions of  $x^{(m)}$ , with the corresponding  $x^{(n-m)}$  fixed in an appropriate way, belong to  $W_{\rho,2}^{(\bar{l})}(R_m)$ , with  $\bar{l} = \mu l$ . Moreover,

$$\left\| \frac{\partial^\alpha f}{\partial x_{m+1}^{r_{m+1}} \dots \partial x_n^{r_n}} \right\|_{W_{\rho,2}^{(\bar{l})}(R_m)} \leq c_7 \|f\|_{W_{\rho, x_{m+1}, \dots, x_n, 2}^{(l, l_{m+1}, \dots, l_n)}(R_n)}, \quad (22)$$

where the constant  $c_7$  does not depend on  $x^{(n-m)}$  or on  $f(x)$ .

The functional space  $W_{x_1, \dots, x_m, \rho, 2}^{(l_1, \dots, l_m, l)}(R_n)$  is defined analogously for nonnegative numbers  $l_1, \dots, l_m, l$ .

**Theorem 8.** Let nonnegative integers  $\alpha$  be given for which

$$\mu = \mu(\alpha) = 1 - 2\alpha/l - (n - m)/2l > 0. \quad (23)$$

For such  $\alpha$ , prescribe on  $R_m$  functions

$$\varphi^{(\alpha)}(x^{(m)}) \in W_{x_1, \dots, x_m, 2}^{(\bar{l}_1, \dots, \bar{l}_m)}(R_m), \quad \bar{l}_i = \mu l_i \quad (i = 1, 2, \dots, m).$$

There exists a function  $\bar{f} \in W_{x_1, \dots, x_m, \rho, 2}^{(l_1, \dots, l_m, l)}(R_n)$  such that

$$\lim_{\rho \rightarrow 0} \left\| \frac{\partial^\alpha \bar{f}(x^{(m)}, x_0^{(n-m)}, \rho, \omega_{n-m})}{(\rho \partial \rho)^\alpha} - \varphi^{(\alpha)}(x^{(m)}) \right\|_{W_{x_1, \dots, x_m, 2}^{(\bar{l}_1, \dots, \bar{l}_m)}(R_m)} = 0 \quad (24)$$

for all admissible  $\alpha$ . Moreover,

$$\|\bar{f}\|_{W_{x_1, \dots, x_m, \rho, 2}^{(l_1, \dots, l_m, l)}(R_n)} \leq c_8 \sum_{\alpha} \|\varphi^{(\alpha)}(x^{(m)})\|_{W_{x_1, \dots, x_m, 2}^{(\bar{l}_1, \dots, \bar{l}_m)}(R_m)}, \quad (25)$$

where the constant  $c_8$  does not depend on  $\varphi^{(\alpha)}(x^{(m)})$ .

The converse assertion is also valid:

**Theorem 9.** Let  $f(x) \in W_{x_1, \dots, x_m, \rho, 2}^{(l_1, \dots, l_m, l)}(R_n)$ , and suppose that for some nonnegative integers  $\alpha$  inequality (23) is satisfied. Then  $\partial^\alpha f / (\rho \partial \rho)^\alpha$ , as functions of  $x^{(m)}$ , with the corresponding  $\rho$  and  $\omega_{n-m}$  fixed in an appropriate way, belong to

$$W_{x_1, \dots, x_m, 2}^{(\bar{l}_1, \dots, \bar{l}_m)}(R_m) \quad \text{with} \quad \bar{l}_i = \mu l_i \quad (i = 1, 2, \dots, m).$$

Moreover, the inequality

$$\left\| \partial^\alpha f / (\rho \partial \rho)^\alpha \Big|_{\rho=0} \right\|_{W_{x_1, \dots, x_m, 2}^{(\bar{l}_1, \dots, \bar{l}_m)}(R_m)} \leq c_9 \|f\|_{W_{x_1, \dots, x_m, \rho, 2}^{(l_1, \dots, l_m, l)}(R_n)} \quad (26)$$

holds, where  $c_9$  does not depend on the function  $f(x)$ .

In conclusion, we note that results analogous to those given above also hold for bounded domains in  $R_n$ .

Received  
8 VI 1962

## CITED LITERATURE

1. L. N. Slobodetskii, *DAN*, **118**, No. 2 (1958).

*Note: Figure translations are in progress. See original paper for figures.*

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