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Abstract

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ON GREEN' S FUNCTIONS OF ARBITRARY (IN PARTICULAR, INTEGER) POWERS OF ELLIPTIC OPERATORS

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1. In ([1]) the properties of the Green' s functions $G(t, x, y)$ of parabolic operators were investigated under various boundary conditions. For definiteness, we shall dwell on the second boundary condition; for notation see ([1]). Identity (5) from ([1]) shows that the operator-function defined by the formula

$$T(t)f(x) = \int_{\Omega} G(t; x, y) f(y)\, dy \tag{1}$$

is a semigroup. It turns out that $T(t)$ is a strongly continuous semigroup ([2]) in any space $L_p(\Omega)$ $(1 \leq p \leq \infty)$ or $C(\overline{\Omega})$, whose norm satisfies the inequalities

$$\|T(t)\|_{L_p(\Omega)} \leq \exp\{-a_0 t\}, \qquad \|T(t)\|_{C(\Omega)} \leq \exp\{-a_0 t\}. \tag{2}$$

We denote by $-A$ the infinitesimal generator of the semigroup $T(t)$ (in some space $L_p(\Omega)$ or $C(\overline{\Omega})$). We shall call the operator A elliptic.

Theorem 1. *Let the function $f(x)$ be continuous on $\overline{\Omega}$ and satisfy some Hölder condition inside Ω. Then the function $u(x) = A^{-1} f(x)$ is continuous on $\overline{\Omega}$, twice continuously differentiable inside Ω, satisfies the equation*

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ik}(x)\frac{\partial u}{\partial x_k}\right] + a(x)u = f(x) \qquad (x \in \Omega) \tag{3}$$

and the boundary condition

$$\lim_{\substack{x\to y \\ x\in N_y}} \sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial u}{\partial x_k}\cos(N_y, x_i) + \sigma(y)u = 0 \qquad (y \in S). \tag{4}$$

If the function $\sigma(y)$ satisfies some Hölder condition, then the function $u(x)$ is once continuously differentiable on $\overline{\Omega}$ and satisfies the boundary condition

$$\sum_{i,k=1}^{n} a_{ik}(y)\frac{\partial u}{\partial y_k}\cos(N_y, y_i) + \sigma(y)u = 0 \qquad (y \in S). \tag{5}$$

Theorem 2. *For any λ with $\operatorname{Re}\lambda \geq 0$ and $1 < p < \infty$, the operator $A + \lambda I$ has a bounded inverse, and*

$$\|[A + \lambda I]^{-1}\|_{L_p(\Omega)} \leq c_p[|\lambda| + 1]^{-1}. \tag{6}$$

Hence, and from (3), it follows that the operator-function $T(t)$ is uniformly differentiable for $t > 0$, and

$$\|T'(t)\|_{L_p(\Omega)} \leq c_p \exp\{-a_0 t\} t^{-1}. \tag{7}$$

2. The estimates (2) make it possible to define arbitrary negative powers of A:

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty T(t) t^{\alpha-1}\, dt. \tag{8}$$

Hence, from (1) it follows:

Theorem 3. For any $\alpha > 0$, the operator $A^{-\alpha}$ is integral, with kernel $G_\alpha(x, y)$ defined by the formula

$$G_\alpha(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty G(t; x, y) t^{\alpha-1}\, dt. \tag{9}$$

We shall call $G_\alpha(x, y)$ the Green's function of the operator A^α. Other approaches to the construction of $G_\alpha(x, y)$ were proposed by M. Riesz ([4]) and V. A. Il'in ([5]). From the properties of $G(t; x, y)$ it follows: $G_\alpha(x, y)$ is symmetric, nonnegative, and fair.

Theorem 4. The estimates

$$G_\alpha(x, y) \leqslant c_\alpha r_{xy}^{2\alpha-n} \qquad (0 < \alpha < n/2); \tag{10}$$

$$G_{n/2}(x, y) \leqslant c_{n/2}\,[\,|\ln r_{xy}| + 1\,]; \tag{11}$$

$$G_\alpha(x, y) \leqslant c_\alpha \qquad (\alpha > n/2) \tag{12}$$

hold. The estimate (12) can be sharpened:

$$G_\alpha(x,y) \leqslant c(\alpha_0)\exp\{\omega\alpha\} \qquad (\alpha \geqslant \alpha_0 > n/2). \tag{13}$$

If λ_0 (see ([1])) is sufficiently large, then it can be shown that $G_\alpha(x,y)$ and the norm of the operator $A^{-\alpha}$ decrease exponentially as $\alpha \to \infty$.

3. **Theorem 5.** For any $\alpha > 0$ and any integer m, the operator $(\ln A)^m A^{-\alpha}$ ([6]) is integral, with symmetric kernel

$$G_{\alpha,(\ln)^m}(x,y) = (-1)^m \int_0^\infty G(t;x,y)\frac{d^m}{d\alpha^m}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]dt. \tag{14}$$

Theorem 6. The estimates

$$|G_{\alpha,(\ln)^m}(x,y)| \leqslant \begin{cases} c_{\alpha,(\ln)^m} r_{xy}^{2\alpha-n}\,[\,|\ln r_{xy}|^m+1\,], & (0<\alpha<n/2);\\ c_{n/2,(\ln)^m}\,[\,|\ln r_{xy}|^{m+1}+1\,]; & \\ c_{\alpha,(\ln)^m}, & (\alpha > n/2). \end{cases} \tag{15}$$

hold.

Theorem 7. Suppose that the function $G_\alpha(x,y)$ and the norm of the operator $A^{-\alpha}$ decrease exponentially as $\alpha \to \infty$. Then, for any $\alpha, \beta > 0$, the operator $A^{-\alpha}(\ln A)^{-\beta}$ is integral, with nonnegative symmetric kernel

$$G_{\alpha,(\ln)^{-\beta}}(x,y) = \frac{1}{\Gamma(\beta)}\int_0^\infty G_{\alpha+s}(x,y)s^{\beta-1}\,ds. \tag{16}$$

Theorem 8. If $0 < \alpha < n/2$, then

$$G_{\alpha,(\ln)^{-\beta}}(x,y) \leqslant c_{\alpha,(\ln)^{-\beta}} r_{xy}^{2\alpha-n}[\,|\ln r_{xy}|^\beta+1\,]^{-1}. \tag{17}$$

If $\alpha = n/2$, then

$$G_{n/2,(\ln)^{-\beta}}(x,y) \leqslant \begin{cases} c_{n/2,(\ln)^{-\beta}}\,[\,|\ln r_{xy}|^{1-\beta}+1\,], & (\beta<1);\\ c_{n/2,(\ln)^{-1}}\,[\,|\ln|\ln r_{xy}||+1\,]; & \\ c_{n/2,(\ln)^{-\beta}}\,[\,|\ln r_{xy}|^{\beta-1}+1\,]^{-1}, & (\beta>1). \end{cases} \tag{18}$$

If $\alpha > n/2$, then

$$G_{\alpha,(\ln)^{-\beta}}(x,y) \leqslant c_{\alpha,(\ln)^{-\beta}}. \tag{19}$$

For the proof one must use formulas (8), (1), and

$$(\ln A)^{-\beta} = \frac{1}{\Gamma(\beta)} \int A^{-s} s^{\beta-1}\, ds. \tag{20}$$

With the help of Theorems 5 and 7 and the semigroup property, it is proved that, for any $\alpha, \beta > 0$, the operator $(\ln A)^{\beta} A^{-\alpha}$ is integral, with symmetric kernel $G_{\alpha,(\ln)^{\beta}}(x, y)$. Theorems 6 and 8 make it possible to establish that, for this ···

the estimates (15) (for $m = \beta$) hold for the kernels. Analogously one can obtain integral representations for the operators $(\ln\ln A)^{\delta}(\ln A)^{\beta} A^{-\alpha}$ for arbitrary δ, β and any $\alpha > 0$, and estimate the kernels of these operators, etc.

4. From the properties of the function $G(t; x, y)$ it follows that all kernels of fractional powers are continuous jointly in the variables for $x, y \in \overline{\Omega}$ and $x \neq y$. If the kernel is bounded, it is also continuous on the diagonal $x = y$. Moreover, the estimates of the paper (1) make it possible to estimate, in a qualified way, the smoothness of these kernels.

Theorem 9. Let $\nu_0 \in (0, 1)$, $\nu \in [0, \nu_0]$. Then

$$r_{xy}^{-\nu}\, |G_\alpha(x, z) - G_\alpha(y, z)| \leq$$

$$\leq c_\alpha(\nu_0) \cdot \begin{cases} r^{2\alpha-n-\nu} & (0 < \alpha < (n+\nu)/2), \\ |\ln r| + 1 & (\alpha = (n+\nu)/2), \\ 1 & (\alpha > (n+\nu)/2), \end{cases} \qquad r = \min\{r_{xz}, r_{yz}\}. \tag{21}$$

For the proof one must use estimates (8), (9) from (1). Analogously one can estimate the Hölder coefficients of the function $G_\alpha(x, y)$ jointly in the variables x and y.

Theorem 10. Let the function $\sigma(y)$ satisfy condition (10) from (1). Then the functions $G_\alpha(x, y)$ are continuously differentiable with respect to x and with respect to y, and jointly in (x, y), for $x, y \in \overline{\Omega}$, $x \neq y$, and

$$\left|\frac{\partial}{\partial x_i} G_\alpha(x, y)\right|, \quad \left|\frac{\partial}{\partial y_i} G_\alpha(x, y)\right| \leq c_\alpha \cdot \begin{cases} r_{xy}^{2\alpha-n-1} & (0 < \alpha < (n+1)/2); \\ |\ln r_{xy}| + 1 & (\alpha = (n+1)/2); \\ 1 & (\alpha > (n+1)/2); \end{cases} \tag{22}$$

$$\left|\frac{\partial^2}{\partial x_i \partial y_k} G_\alpha(x, y)\right| \leq c_\alpha \cdot \begin{cases} r_{xy}^{2\alpha-n-2} & (0 < \alpha < (n+2)/2); \\ |\ln r_{xy}| + 1 & (\alpha = (n+2)/2); \\ 1 & (\alpha > (n+2)/2); \end{cases} \tag{23}$$

Theorem 11. Let $\nu_0 \in (0, \min\{\lambda, h\})$, $\nu \in [0, \nu_0]$. Then

$$r_{xy}^{-\nu} \left| \partial G_\alpha(x,z)/\partial x_i - \partial G_\alpha(y,z)/\partial y_i \right| \leq$$

$$\leq c_\alpha(\nu_0) \cdot \begin{cases} r^{2\alpha-n-1-\nu} & (0 < \alpha < (n+1+\nu)/2); \\ |\ln r| + 1 & \alpha = (n+1+\nu)/2; \\ 1 & (\alpha > (n+1+\nu)/2); \end{cases} \qquad r = \min\{r_{xz}, r_{yz}\}. \quad (24)$$

For the proof one must use estimates (11), (12), (13), (14), (16), (17) from (1). Analogously one can estimate the Hölder coefficients of the functions $\partial G_\alpha(x,y)/\partial x_i$, $\partial^2 G_\alpha(x,y)/\partial x_i \partial y_k$ jointly in the variables x and y. In the general case (i.e. when the function $\sigma(y)$ is only continuous), to estimate the derivatives of the functions $G_\alpha(x,y)$ one must use the identity

$$G_\alpha(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^{t_0} G(t;x,y) t^{\alpha-1}\, dt + \int_{\Omega_z} G(t_0;x,z) \left[\frac{1}{\Gamma(\alpha)} \int_{t_0}^{\infty} G(t_0;z,y) t^{\alpha-1}\, dt \right] dz. \quad (25)$$

5. From the properties of the function $G(t;x,y)$ it follows that all functions $G_\alpha(x,y)$ for $x \neq y$ satisfy the boundary condition (4). If, however, $\sigma(y)$ is smooth, then $G_\alpha(x,y)$ satisfy (5).

Theorem 12. The estimates

$$\left| \frac{d}{dT_{p_x}} G_\alpha(x,y) \right| \leq c_\alpha \cdot \begin{cases} r_{xy}^{2\alpha-n-1} & (0 < \alpha < (n+1)/2); \\ |\ln r_{xy}| + 1 & (\alpha = (n+1)/2); \\ 1 & (\alpha > (n+1)/2); \end{cases} \quad (26)$$

hold. If $x, y \in S$, then

$$\left| \frac{d}{dT_x} G_\alpha(x,y) \right| \leq c_\alpha \cdot \begin{cases} r_{xy}^{2\alpha-n-1+\lambda} & (0 < \alpha < (n+1-\lambda)/2); \\ |\ln r_{xy}| + 1 & (\alpha = (n+1-\lambda)/2); \\ 1 & (\alpha > (n+1-\lambda)/2). \end{cases} \quad (27)$$

Theorem 13. Let $\nu_0 \in (0, \lambda)$, $\nu \in [0, \nu_0]$. Then

$$r_{xy}^{-\nu} \left| dG_\alpha(x,z)/dT_{p_x} - dG_\alpha(y,z)/dT_{p_y} \right| \leq$$

$$\leq c_\alpha(\nu_0)\cdot\begin{cases} r^{2\alpha-n-1-\nu} & (0<\alpha<(n+1+\nu)/2),\\ |\ln r|+1 & (\alpha=(n+1+\nu)/2),\\ 1 & (\alpha>(n+1+\nu)/2);\end{cases} \qquad r=\min\{r_{xz},r_{yz}\}. \quad (28)$$

If $x, y, z \in S$, then

$$r_{xy}^{-\nu}\,\big|\,dG_\alpha(x,z)/dT_x - d/G_\alpha(y,z)/dT_y\,\big| \leq$$

$$\leq c_\alpha(\nu_0)\cdot\begin{cases} r^{2\alpha-n-1-\nu+\lambda} & (0<\alpha<(n+1+\nu-\lambda)/2),\\ |\ln r|+1 & (\alpha=(n+1+\nu-\lambda)/2),\\ 1 & (\alpha>(n+1+\nu-\lambda)/2);\end{cases} \qquad r=\min\{r_{xz},r_{yz}\}. \quad (29)$$

6. The known theorems of S. L. Sobolev on integrals of potential type and estimates of the kernels of fractional powers make it possible to determine from which spaces into which spaces the operators $A^{-\alpha}$ act.

Theorem 14. Let $0<\alpha<n/2$ and $1<p<\infty$. 1) Let $p<n/2\alpha$. Then the operator $A^{-\alpha}$ acts from $L_p(\Omega)$ into $L_q(\Omega)$, where $q=np/(n-2\alpha p)$, and is bounded. 2) Suppose, in addition, that $p>(n-s)/2\alpha$ $(s=1,2,\dots,n-1)$. Then the operator $A^{-\alpha}$ acts from $L_p(\Omega)$ into $L_q(\Omega_s)$, where $q=sp/(n-(2\alpha p))$, and Ω_s is any smooth s-dimensional manifold contained in Ω. 3) Let $n/2\alpha<p<n/(2\alpha-1)$, $\alpha>1/2$. Then the operator $A^{-\alpha}$ acts from $L_p(\Omega)$ into any $C_{0,\beta}(\Omega)$, where $\beta<2\alpha-n/p$, and is bounded.

Theorem 15. Let $1/2<\alpha<(n+1)/2$, $1<p<\infty$. 1) Let $p<n/(2\alpha-1)$. Then the operator $A^{-\alpha}$ acts from $L_p(\Omega)$ into $W_q^1(\Omega)$, where $q=np/[\,n-(2\alpha-1)p\,]$, and is bounded. 2) Suppose, in addition, that $p>(n-s)/(2\alpha-1)$ $(s=1,2,\dots,n-1)$ and $\sigma(y)$ satisfies condition (10) from (1). Then the operator $A^{-\alpha}$ acts from $L_p(\Omega)$ into $W_q^1(\Omega_s)$, where $q=sp/[\,n-(2\alpha-1)p\,]$, and is bounded. 3) Suppose the preceding assumptions are fulfilled, let $\alpha>\frac{1}{2}[\,1+\min\{\lambda,h\}\,]$, and let

$$n/(2\alpha-1)<p<n/(2\alpha-1-\min\{\lambda,h\}).$$

Then the operator $A^{-\alpha}$ acts from $L_p(\Omega)$ into any $C_{1,\nu}(\Omega)$, where $\nu<2\alpha-1-n/p$, and is bounded.

From a result of M. A. Krasnosel' skii and Ya. B. Rutitskii ([7]) and estimate (11) it follows:

Theorem 16. The operator $A^{-n/2}$ acts from $L^*_\Phi(\Omega)$ into $L^*_\Psi(\Omega)$ and is bounded. Here $L^*_\Phi(\Omega)$, $L^*_\Psi(\Omega)$ are Orlicz spaces constructed respectively from the functions

$$\Phi(u)=(1+|u|)\ln(1+|u|)-|u|, \qquad \Psi(u)=\exp\{|u|\}-|u|-1.$$

7. From the results of item 5 it follows that every function $u(x) \in D(A^\alpha)$ $(\alpha > 1/2)$, for $p > n/(2\alpha - 1)$, satisfies the boundary condition (4). If $\sigma(y)$ is smooth, then (5) is fulfilled.

8. From Theorem 15 it follows that for any $\varepsilon \in (0, 1/2)$

$$D(A^{1/2+\varepsilon}) \subset W_p^1(\Omega) \subset D(A^{1/2-\varepsilon}) \;* . \tag{30}$$

9. All the preceding results are valid also in the case of the first boundary-value problem. We note that the kernels of the fractional powers $A^{-\alpha}$ of this problem do not exceed the corresponding kernels of the second boundary-value problem.

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* In the case $p = 2$ this proposition is valid for $\varepsilon = 0$.

Note: Figure translations are in progress. See original paper for figures.