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Abstract

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ON THE THEORY OF PERIODIC AND LIMIT-PERIODIC JACOBI MATRICES

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Periodic, as well as limit-periodic, Jacobi matrices and the continued fractions associated with them have for a long time been the subject of a number of investigations (see ([1]) and the bibliography cited there). In the present note we develop one algebraic approach to such matrices, which, in combination with operator-theoretic methods ([2],[3]), leads to some new results.

1. By a generalized Jacobi matrix of order $2m$ we shall mean an infinite matrix
$$A = \|a_{jk}\| \quad (j, k = \pm 0, 1, 2, ...)$$
with complex elements satisfying the conditions $a_{jk} = 0$ for $|j-k| > m$, $a_{jk} \neq 0$ for $|j-k| = m$. For $m = 1$ the generalized Jacobi matrix is the ordinary Jacobi matrix. The following theorem is a discrete analogue of one theorem of Burchnall and Chaundy ([4]) on linear differential expressions.

Theorem 1. *If generalized Jacobi matrices A and B, of orders $2m$ and $2n$, respectively, commute, then they satisfy an algebraic relation $D(A, B) = 0$ of degree $2n$ with respect to A and of degree $2m$ with respect to B.*

Proof is analogous to that given in ([4]) for the continuous case.

Denote by E_n the matrix all of whose elements, with the exception of
$$e_{k,k-n} = e_{k,k+n} = 1 \quad (k = \pm 0, 1, 2, ...),$$
are equal to zero. It is easy to see that the spectrum $S(E_n)$ is the interval $-2 \leq \mu \leq 2$ and each point $\lambda \in S(E_n)$ has multiplicity $2n$. If a generalized Jacobi matrix A has period n, then it obviously commutes with E_n, so that $D(A, E_n) = 0$. In the simplest case of an ordinary n-periodic complex-symmetric Jacobi matrix
$$T = \|t_{jk}\| \quad (t_{jk} = t_{kj};\ j, k = \pm 0, 1, 2, ...)$$
the polynomial $D(\lambda, \mu)$ is an exact square, so that the following holds:

Theorem 2. *Every complex-symmetric n-periodic Jacobi matrix T satisfies an algebraic equation of the n-th degree*
$$P(T) = E_n. \tag{1}$$

If the matrix T is real, then the coefficients of the polynomial $P(\lambda)$ are also real. Since the spectrum $S(E_n)$ is known, relation (1) makes it possible to characterize completely the spectrum $S(T)$. In particular, in the case of a real matrix one obtains the known result on the structure of the spectrum ([1]).

Theorem 3. *The spectrum $S(T)$ of a complex-symmetric n-periodic Jacobi matrix T coincides with the full λ-preimage Γ of the interval $-2 \le \mu \le 2$ under the mapping*

$$P(\lambda) = \mu \tag{2}$$

and, consequently, consists of n algebraic arcs, which in special cases may have common endpoints.

If the matrix T is real, then for $-2 \le \mu \le 2$ all λ—the roots of equation (2)—are real, so that $S(T)$ is a system of n intervals, which in special cases may have common endpoints. Moreover, each point $\lambda \in S(T)$ has multiplicity two.

Proof. The relation $S(T) \subset \Gamma$ follows from (1). If the matrix T is real, then the multiplicity of $S(T)$ does not exceed two, so that the multiplicity of the spectrum of the left-hand side of (1) is not greater than $2n$. Since, at the same time, the multiplicity of each point of the spectrum $S(E_n)$ is $2n$, all assertions of the theorem pertaining to the real case follow from equality (1).

In the general case of a complex matrix T, it is not difficult, following the well-known Floquet scheme, to establish that a necessary and sufficient condition for the boundedness of at least one of the solutions $y = \{y_k\}_{k=-\infty}^{\infty}$ of the finite-difference equation

$$t_{k,k-1} y_{k-1} + t_{k,k} y_k + t_{k,k+1} y_{k+1} = \lambda y_k \tag{3}$$

for a given complex λ is the inequality $-2 \le P(\lambda) \le 2$, so that the set Γ is a system of n curvilinear stability zones of equation (3). Let now $\lambda \in \Gamma$. If in this case $\{y_k\}_{k=-\infty}^{\infty} \in l_2$, then, evidently, the number λ would be an eigenvalue of T. If, however, $\{y_k\}_{k=-\infty}^{\infty} \bar{\in} l_2$, then

$$\sum_{k=-\infty}^{\infty} |y_k|^2 = \infty$$

and for the sequence of vectors z_N with coordinates

$$z_{N,k} = \begin{cases} y_k, & (|k| \le N), \\ 0, & (|k| > N) \end{cases}$$

we shall have

$$\lim_{N\to\infty} \frac{\|T z_N - \lambda z_N\|}{\|z_N\|} = 0,$$

so that $\lambda \in S(T)$. Thus, $\Gamma \subset S(T)$, and the theorem is proved.

2. In the real case we number the gaps in $S(T)$ by the indices $1, 2, \dots, n+1$, beginning with the gap extending from $-\infty$, and take $\widetilde{T}$ to be obtained by adding a completely continuous real diagonal matrix K. By the well-known theorem of H. Weyl on completely continuous perturbations, the spectrum $S(T+K)$ will consist of the system of intervals described in Theorem 3 and of a bounded set of eigenvalues, which can accumulate only at the ends of the gaps. Evidently, the corresponding perturbation

$$Q = P(T+K) - P(T)$$

of the matrix $P(T)$ is a completely continuous generalized Jacobi matrix of order $2n-2$. Thus, the polynomial $P(\lambda)$ transforms the limit-periodic matrix $T+K$ into the limit-constant matrix

$$P(T+K) = E_n + Q. \tag{4}$$

In view of relation (4), the conditions for the finiteness or infiniteness of the set of points of the spectrum $S(T+K)$ lying in the union of all odd or even gaps of the spectrum $S(T)$ coincide with the conditions for the finiteness or infiniteness of the set of points of the spectrum of the limit-constant matrix $E_n + Q$, situated to the left of the point $\mu = -2$ or to the right of the point $\mu = 2$. The determination of these latter conditions is carried out as in ([3]) and leads to the following results.

Denote by σ_r the sum of the absolute values of all elements of the r-th row of the matrix Q, except for its diagonal element q_r, and put

$$\omega'_r = \sigma_r - q_r, \qquad \omega''_r = \sigma_r + q_r.$$

Further, by S' and S'' denote the set of spectral points introduced by the perturbation K into the union of all odd and, respectively, even gaps of the spectrum $S(T)$. Without loss of generality, we shall assume that

$$(-1)^n t_{12} \cdot t_{23} \cdot \dots \cdot t_{n,n+1} > 0.$$

Theorem 4. If

$$\limsup_{|r|\to\infty} r^2 \omega'_r < \frac{n^2}{4},$$

then S' is finite. If

$$\limsup_{|r|\to\infty} r^2 \omega''_r < \frac{n^2}{4},$$

then S'' is finite. If one of the $2n$ series

$$\sum_{r=0}^{\pm\infty} q_{j+nr} \quad (j = 1, 2, \dots, n) \tag{5}$$

diverges to $+\infty$, then S' is infinite. If one of the $2n$ series (5) diverges to $-\infty$, then S'' is infinite.

Using Theorem 4, one can obtain conditions expressed only in terms of the elements k_r of the perturbation K. Thus, for example, when $k_r = o\left(\frac{1}{r^2}\right)$, the set $S' + S''$ will be finite. In the particular case $n = 2$, assuming for definiteness that $t_{12} - t_{10} > 0$, we arrive in this way at the following result.

Theorem 5. If one of the two series

$$\sum_{-\infty}^{+\infty} k_{2r-1}$$

consists only of positive terms and diverges, then the perturbation K introduces into the internal gap of the 2-periodic matrix T an infinite set of eigenvalues. If, however, one of the two series

$$\sum_{-\infty}^{+\infty} k_{2r}$$

consists only of positive terms and diverges, then the perturbation K introduces an infinite set of eigenvalues into the union of the external gaps.

An analogous result holds for arbitrary n. In those cases when almost all $k_r \geq 0$ (or ≤ 0), the spectrum introduced by the perturbation K cannot accumulate at the left (right) ends of the gaps.

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References Cited

1. Ya. L. Geronimus, *Izv. AN SSSR, ser. matem.*, **5**, 203 (1943).

2. I. M. Glazman, *DAN*, **118**, 423 (1958).

3. P. B. Naĭman, *Izv. Vyssh. uchebn. zaved.*, No. 1(8), 129 (1959).

4. E. L. Ince, *Ordinary Differential Equations*, 1939.