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Abstract

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MATHEMATICS

G. P. TOLSTOV

DERIVATIVE AND INTEGRAL FROM A GENERAL POINT OF VIEW

(Presented by Academician P. S. Aleksandrov on 9 X 1961)

Let X be an abstract space (i.e., a set of arbitrary elements); let K be a nonempty class of subsets of the space X (we shall not consider other sets below); let $L = \{F(E)\}$ be a class of set functions defined for $E \in K$; and let $M = \{f(x)\}$ be a class of point functions defined on the whole space X.

Suppose that the class L is mapped onto the class M in such a way that to each $F(E) \in L$ there corresponds one, or some family $[f(x)]_F$, of functions from M.

We shall agree to call such a mapping **differentiation**; any $f(x)$ from the family $[f(x)]_F$ a **derivative** of $F(E)$; and $F(E)$ an **antiderivative**, or **integral**, of $f(x)$, if the following requirements are satisfied.

I. The mapping is linear in the following sense: if $f(x)$ is a derivative of $F(E)$, and $g(x)$ is a derivative of $G(E)$, then for any real constants α and β the function $\alpha f(x) + \beta g(x)$ is a derivative of $\alpha F(E) + \beta G(E)$.

II. The function $f(x) \equiv 1$ is a derivative of some function $\mu(E)$, i.e., certainly $1 \in M$.

III. If $f(x)$ is a derivative of $F(E)$ and $f(x) \geq 0$ on $\mathcal{E}$, then $F(\mathcal{E}) \geq 0$*.

From I and III it follows without difficulty that the values of the derivative on a certain set determine the value of the antiderivative on that set **completely rigidly**. The function $\mu(E)$ (see II) will be called a **measure**.

We shall agree to write the fact that $f(x)$ is a derivative of $F(E)$ as

$$\frac{dF}{d\mu} = f(x)$$

or as

$$\int_E f(x)\, d\mu = F(E); \tag{1}$$

for fixed E, the number (1) will also be called the **integral of $f(x)$ over the set** E. In view of what was said above, the symbol $\frac{dF}{d\mu}$ is, in general, not uniquely determined; the integral (1) is uniquely determined. The words "a function is integrable on the set E" will mean that this function on E coincides with one of the functions of the class M.

To the terms **differentiation**, **derivative**, **antiderivative**, **integral**, **measure**, used in our very general sense, in order to avoid any confusion it is sometimes convenient to attach the epithet **formal**: formal differentiation, formal integral, etc.

The following properties of the formal integral are valid:

$$1^\circ. \int_E d\mu = \mu(\mathcal{E}).$$

* The consistency of requirements I—III is sufficiently obvious.

2°.

$$\int_E [\alpha f(x) + \beta g(x)]\, d\mu = \alpha \int_E f(x)\, d\mu + \beta \int_E g(x)\, d\mu, \quad \alpha = \text{const},\ \beta = \text{const}$$

(from the existence of the integrals on the right-hand side follows the existence of the integral on the left and the equality).

3°. If $f(x)$ is integrable on $\mathcal{E}$, then it is integrable also on $E \subset \mathcal{E}$.

4°. If $f(x)$ is integrable and nonnegative on $\mathcal{E}$, then

$$\int_E f(x)\, d\mu \geqslant 0.$$

These properties entail many consequences; in particular:

A. If $f(x)$ and $g(x)$ are integrable on $\mathcal{E}$ and $f(x) \geqslant g(x)$, then

$$\int_E f(x)\, d\mu \geqslant \int_E g(x)\, d\mu.$$

B. If $f(x)$ is integrable and bounded on $\mathcal{E}$, then: a)

$$\left| \int_E f(x)\, d\mu \right| \leqslant M \cdot \mu(\mathcal{E}), \quad M = \text{const}, \quad |f(x)| \leqslant M;$$

b) for $E \subset \mathcal{E}$ the integral

$$\int_E f(x)\, d\mu$$

is an absolutely continuous function of the set; c) from $\mu(\mathcal{E}) = 0$ it follows that

$$\int_E f(x)\,d\mu = 0.$$

We note that in b) and c) the requirement of boundedness of $f(x)$ is essential.

C. If $f(x)$ and $|f(x)|$ are integrable on $\mathcal{E}$, then

$$\left|\int_E f(x)\,d\mu\right| < \int_E |f(x)|\,d\mu.$$

D. Let $\{f_n(x)\}$ be a sequence of integrable functions uniformly convergent on $\mathcal{E}$. Then: a) there always exists a finite limit

$$\lim_{n\to\infty}\int_E f_n(x)\,d\mu;$$

b) if $f_n(x) \to f(x)$ and $f(x)$ is integrable on $\mathcal{E}$, then

$$\lim_{n\to\infty}\int_E f_n(x)\,d\mu = \int_E f(x)\,d\mu.$$

E. Let X be a metric space, $x_0 \in X$, and let the class K contain sets contracting to the point x_0. If, moreover, the integrable function $f(x)$ is continuous at the point x_0, then

$$\int_E f(x)\,d\mu = f(x_0)\mu(E) + \alpha\mu(E),$$

where $\alpha \to 0$ when E contracts to x_0. In other words, the set function

$$\int_E f(x)\,d\mu$$

is differentiable at the point x_0 with respect to the function $\mu(E)$, and its differential has the value $f(x_0)\mu(E)$.

5°. Fix some function $\varphi(x) \in M$, $\varphi(x) > 0$, put

$$\nu(E) = \int_E \varphi(x)\,d\mu \tag{2}$$

and introduce for the class L a new operation of differentiation in the following way. As the derivative of the function $F(E)$ we take any of the functions of the family

$$\left[\frac{f(x)}{\varphi(x)}\right],$$

where $f(x) \in [(f(x)]_F$. The primitive for 1, i.e. the measure under the new conditions, turns out to be the function (2). For the new formal integral

$$\int_E \frac{f(x)}{\varphi(x)}\, d\nu = \int_E f(x)\, d\mu$$

this is the rule for transforming the measure.

6°. Consider some mapping $x = \varphi(t)$ of an arbitrary space T onto X. In the space T we introduce the operation of differentiation as follows. To each set $E \in K$ we assign its inverse image $\mathcal{E} = \varphi^{-1}(E)$ (not necessarily complete) and put $K_T = \{\mathcal{E}\}$, $L_T = \{F(\varphi(\mathcal{E}))\}$, $M_T = \{f(\varphi(t))\}^*$. As the derivative of $F(\varphi(\mathcal{E}))$ we assign any of the functions of the family $[f(\varphi(t))]$, where $f(x) \in [f(x)]_F$. The measure is the function $\mu(\varphi(\mathcal{E}))$. For the corresponding integral:

$$\int_E f(x)\, d\mu = \int_{\mathcal{E}} f(\varphi(t))\, d(\mu\varphi)$$

the first rule for change of variable.

7°. Let $t = \psi(x)$ be some mapping of the space X onto the space T. In T we introduce the operation of differentiation as follows. To each set $E \in K$ we assign its image $\mathcal{E} = \psi(E)$ and put $K_T = \{\mathcal{E}\}$. By $\psi^{-1}(\mathcal{E})$ denote one of the inverse images of the set $\mathcal{E}$ that is contained in K. We now consider the totality of those functions from M which are representable in the form $f(x) = g(\psi(x))$ (such, for example, is the function $f(x) \equiv 1$) and put $M_T = \{g(t)\}$. To each $g(t)$, as an antiderivative, we assign the function $F(\psi^{-1}(\mathcal{E}))$, where $F(E)$—as an antiderivative—we assign the function $F(\psi^{-1}(\mathcal{E}))$, where $F(E)$ is an antiderivative for the corresponding $f(x)$. The measure is the function $\mu(\psi^{-1}(\mathcal{E}))$. For the corresponding integral

$$\int_{\psi^{-1}(\mathcal{E})} g(\psi(x))\, d\mu = \int_{\mathcal{E}} g(t)\, d(\mu\psi^{-1})$$

the second rule for change of variable.

Despite the formal integral' s possessing the enumerated "good" properties, it nevertheless **may not be additive** (there are corresponding examples). And what happens if one is dealing with a variety of integral that has the property of additivity?

8°. Let the integral be finitely additive, and let $f(x)$ be integrable on $\mathcal{E}$.

a) If $\mathcal{E}$ is a metric space and $f(x)$ is integrable on $\mathcal{E}$ in the Riemann-Stieltjes sense with respect to the measure μ, then its integral in this sense coincides with the formal integral.

b) If $f(x)$ is bounded and μ-measurable on $\mathcal{E}$, then it is integrable on $\mathcal{E}$ in the Lebesgue sense with respect to the measure μ**, and its integral in this sense again coincides with the formal integral.

9°. Let the integral be countably additive, and let $f(x)$ be integrable on $\mathcal{E}$. If, in addition, $f(x)$ is μ-measurable on $\mathcal{E}$, then it is integrable on $\mathcal{E}$ in the Lebesgue sense with respect to the measure μ, and the values of the corresponding integrals again coincide.

In conclusion, let us note the following. Requirements I-III are in a certain sense minimal in order for it to be natural to speak of operations of differentiation or integration. Appending to them **new** requirements leads to more special kinds of the operations mentioned. Along this path one can obtain integrals which, in their properties, are quite similar to the classical integrals–the integral of an exact derivative ("Dugamel integral"), the Denjoy integral, and the Lebesgue integral.

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* K_T, L_T, M_T are the classes K, L, M for the space T.

** Here μ-measurability of a function is understood more broadly than usual: only μ-measurability is required of the sets occurring in Lebesgue integral sums, without any restrictions on the structure of the class of all μ-measurable sets. Lebesgue integrability is understood just as broadly: only the existence of the limit of Lebesgue integral sums is required.

Note: Figure translations are in progress. See original paper for figures.