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Abstract

Full Text

MATHEMATICS

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ON THE DEPENDENCE BETWEEN THE ORDER OF GROWTH OF ORTHONORMAL POLYNOMIALS AND THE CHARACTER OF THE MASS DISTRIBUTION

(Presented by Academician S. N. Bernstein on 9 IV 1962)

In paper (1) we considered certain conditions for boundedness of a system of polynomials $\{\varphi_n(z)\}_0^\infty$, orthonormal on the unit circle $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with respect to the mass distribution $d\sigma(\theta)$; in the present note we shall consider the question of the order of growth of these polynomials as a function of the character of the zeros of the function $\sigma'(\theta)$.

1. Theorem 1. If $a(\delta)$ is the modulus of increase of the function $\sigma(\theta)$ on the interval $[0, 2\pi]$ (see (4))

$$a(\delta) = \inf_{\theta} \int_{\theta}^{\theta+\delta} d\sigma(\theta), \quad \theta, \theta + \delta \in [0, 2\pi], \quad \delta > 0, \quad (1)$$

then the inequality

$$K_n(\theta) a\left(\frac{1}{n}\right) \leq 8\pi, \quad K_n(\theta) = \sum_{s=0}^n |\varphi_s(e^{i\theta})|^2, \quad \theta \in [0, 2\pi] \quad (n = 1, 2, \dots) \quad (2)$$

holds. Hence, in addition to the obvious inequality

$$|\varphi_n(e^{i\theta})| \sqrt{a\left(\frac{1}{n}\right)} \leq 2\sqrt{2\pi}, \quad \theta \in [0, 2\pi] \quad (n = 1, 2, \dots), \quad (3)$$

there follows, under the additional condition $\lg \sigma'(\theta) \in L_1$, the stronger result (see (2), § 3.5)

$$\lim_{n \rightarrow \infty} \left\{ |\varphi_n(e^{i\theta})| \sqrt{a\left(\frac{1}{n}\right)} \right\} = 0, \quad \theta \in [0, 2\pi]. \quad (4)$$

Theorem 2. If $s \geq 1$ is an integer and if we introduce the notation

$$M_n = \max_{\theta \in e_n} |\varphi_n(e^{i\theta})|, \quad \theta \in e_n = \left[\theta_0 + \frac{1}{2(n-s)}, \theta_0 + \frac{1}{2(n-s)} \right], \quad n > s,$$

$$\mu(h) = \frac{\sigma(\theta_0 + h) - \sigma(\theta_0 - h)}{2h}, \quad h > 0, \quad (5)$$

then the inequality* holds

$$M_n^2 \mu_1 \left(\frac{1}{n} \right) \geq 1, \quad \mu_1(x) = x^{2s-1} \left\{ C_1 + C_2 \int_x^\pi \frac{\mu(t) dt}{t^{2s}} \right\}. \quad (6)$$

* C, C_1, C_2, \dots are constants independent of n .

Hence, as a special case, there follows a theorem supplementing Theorem 1 of our note ⁽¹⁾:

Theorem 3. Suppose that at the point θ_0 there exists a generalized symmetric derivative

$$\sigma^{(1)}(\theta_0) = \lim_{h \rightarrow 0} \mu(h);$$

if there exists a subsequence $\{\varphi_{n_i}(z)\}$, uniformly bounded in a neighborhood of the point θ_0 ,

$$|\varphi_{n_i}(e^{i\theta})| \leq C, \quad \theta_0 - \varepsilon \leq \theta \leq \theta_0 + \varepsilon, \quad \varepsilon > 0, \quad n_i > \frac{1}{2\varepsilon} + 1, \quad (7)$$

then $\sigma^{(1)}(\theta_0) > 0$.

2. In Theorems 1 and 2 there occur certain structural characteristics of the function $\sigma(\theta)$; the quantity $\mu(h)$ is a purely local characteristic, since h may be taken arbitrarily small; the quantity $a(\delta)$, i.e. the modulus of increase on the whole interval $[0, 2\pi]$, is not a local characteristic; if, however, in Theorem 1 one considers the modulus of increase only on some interior interval, then in (2) one must replace $\frac{1}{n}$ by $\frac{1}{n^2}$.

Theorems 1 and 2 make it possible to establish a certain dependence between the order of growth of the orthonormal system and the character of the zeros of the function $\sigma'(\theta)$.

Theorem 4. 1) Suppose that all zeros $\{\theta_i\}_1^m$ ($m < \infty$) of the function $\sigma'(\theta)$ on the interval $[0, 2\pi]$ are not higher than of logarithmic order, i.e. there exist constants $C, \gamma > 0$ such that in a neighborhood of each of the zeros we have

$$\sigma'(\theta) \geq C |\lg |\theta - \theta_i||^{-\gamma}, \quad \sigma'(\theta_i) = 0 \quad (i = 1, 2, \dots, m); \quad (8)$$

then on the whole interval $[0, 2\pi]$ the inequality

$$|\varphi_n(e^{i\theta})| \leq C_1(\lg n)^{\gamma/2} \quad (n = 2, 3, \dots) \quad (9)$$

holds.

2) If in a neighborhood of some point θ_0 we have

$$\mu(h) \leq C_2 \left(\lg \frac{1}{h} \right)^{-\gamma}, \quad C_2, \gamma > 0, \quad (10)$$

then $M_n \geq C_3(\lg n)^{\gamma/2}$, $n = 2, 3, \dots$; in particular, if in this neighborhood the function $\sigma(\theta)$ is differentiable and $\sigma'(\theta)$ has at the point θ_0 a zero higher than of logarithmic order, then

$$\overline{\lim}_{n \rightarrow \infty} \left\{ \frac{\lg M_n}{\lg \lg n} \right\} = \infty. \quad (11)$$

Theorem 5. 1) Suppose that all zeros of the function $\{\theta_i\}_1^m$ on the interval $[0, 2\pi]$ are not higher than of algebraic order, i.e. there exist constants $C, \gamma > 0$ such that in a neighborhood of each of the zeros we have

$$\sigma'(\theta) \geq C|\theta - \theta_i|^\gamma, \quad \sigma'(\theta_i) = 0 \quad (i = 1, 2, \dots, m); \quad (12)$$

then on the whole interval $[0, 2\pi]$ the inequality

$$|\varphi_n(e^{i\theta})| \leq C_1 n^{\gamma/2} \quad (n = 1, 2, \dots) \quad (13)$$

holds.

2) If in a neighborhood of some point θ_0 we have

$$\mu(h) \leq C_2 h^\gamma, \quad (14)$$

then $M_n \geq C_3 n^{\gamma/2}$, $n = 2, 3, \dots$; in particular, if in this neighborhood the function $\sigma(\theta)$ is differentiable and $\sigma'(\theta)$ has at the point θ_0 a zero higher than of algebraic order, then

$$\overline{\lim}_{n \rightarrow \infty} \frac{\lg M_n}{\lg n} = \infty. \quad (15)$$

3. Theorems 4 and 5, as well as the estimates obtained earlier by us (see ⁽²⁾, § 8.2; ⁽³⁾), make it possible to draw some conclusions about the nature of the zeros of the function $\sigma'(\theta)$ on the whole interval $[0, 2\pi]$, if the order of growth of the orthonormal system at one point is known; these results are summarized in the table:

Order of growth at the given point θ_0	Nature of the zeros of the function $\sigma'(\theta)$ on the interval $[0, 2\pi]$
$\overline{\lim}_{n \rightarrow \infty} \left\{ \frac{\lg \varphi_n(e^{i\theta_0}) }{\lg \lg n} \right\} = \infty$	There are zeros above logarithmic order
$\overline{\lim}_{n \rightarrow \infty} \left\{ \frac{\lg \varphi_n(e^{i\theta_0}) }{\lg n} \right\} = \infty$	There are zeros above algebraic order
$\overline{\lim}_{n \rightarrow \infty} \left\{ \frac{\lg \varphi_n(e^{i\theta_0}) }{n} \right\} > 0$	$\sigma'(\theta) = 0, \theta \in E, \text{mes } E > 0$
$\overline{\lim}_{n \rightarrow \infty} \left\{ \frac{\lg \varphi_n(e^{i\theta_0}) }{\sqrt{n}} \right\} = \infty$	$\lg \sigma'(\theta) \notin L_1(0, 2\pi)$

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Note: Figure translations are in progress. See original paper for figures.

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