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Abstract

Full Text

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SPACES $L_p^r(\Omega)$. EXTENSION AND EMBEDDING THEOREMS

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MATHEMATICS

1. In the preceding note (2), proceeding from the concept of a generalized Liouville derivative, we showed how one can extend the $W_p^{(l)}$-classification of the function spaces of S. L. Sobolev (1) to nonintegral indices of differentiation. In the case of a bounded domain $\Omega \subset E_n$, the corresponding direct construction encounters considerable difficulties. However, using the equivalence of the classes that interest us with classes of functions representable by Bessel potentials (see (2)) and the recent results of E. Stein (3), one can reduce the matter to extending functions from Ω to E_n.* In the present note we, first, carry out (for domains Ω with sufficiently smooth boundary) the indicated extension and, on its basis, come to the consideration of the spaces $L_p^r(\Omega)$; secondly, we obtain a further embedding theorem for the spaces $L_p^r(E_n)$. Naturally, the embedding theorems obtained for $L_p^r(E_n)$ also hold in $L_p^r(\Omega)$.

Definition. A function $f(x) = f(x_1, \dots, x_n)$ belongs to the space $L_p^r(\Omega)$, $2 \le p < \infty$, $r \ge 0$, if it belongs to the Sobolev space $W_p^{([r])}(\Omega)$, and, for nonintegral r, also possesses finite integrals

$$\Lambda_p^r(f^{[r]}, \Omega) = \left\{ \int_\Omega dy \left(\int_\Omega \frac{|f^{[r]}(x) - f^{[r]}(y)|^2}{|x-y|^{\,n+2(r-[r])}} \, dx \right)^{p/2} \right\}^{1/p}, \tag{1}$$

where $f^{[r]}(x)$ is any of the partial derivatives of order $[r]$ of the function f.

With the introduction of the norm

$$\|f\|_{L_p^r(\Omega)} = \begin{cases} \|f\|_{W_p^r(\Omega)}, & \text{for integral } r \ge 0, \\ \|f\|_{W_p^{[r]}} + \sum\limits_{[r]} \Lambda_p^r(f^{[r]}, \Omega), & \text{for nonintegral } r > 0, \end{cases} \tag{2}$$

the space $L_p^r(\Omega)$ becomes a Banach space.

Let us first prove that the present definition of $L_p^r(\Omega)$ for $\Omega = E_n$ coincides with the definition used in ([2]). Namely, there it was assumed that $f(x) \in L_p^r(E_n)$ if**

$$f(x) = c_{n,r} \int_{E_n} |x-t|^{(r-n)/2} K_{\frac{n-r}{2}}(|x-t|)\,\varphi(t)\,dt, \qquad \varphi(t) \in L_p(E_n) \quad (3)$$

and as the norm of f in $L_p^r(E_n)$ the quantity $\|\varphi\|_{L_p(E_n)}$ was taken.

Theorem 1. *For $2 \le p < \infty$, a function $f(x)$ is representable in the form (3) if and only if the inequalities*

$$B_{p,r}\|\varphi\|_{L_p(E_n)} \le \|f\|_{L_p^r(\Omega)} \le A_{p,r}\|\varphi\|_{L_p(E_n)} \qquad (\Omega = E_n!) \quad (4)$$

hold.

* In doing so we take into account the experience of constructing the $H_p^{(r)}$-classes of S. M. Nikol' skii ([4]) and the generalized (after L. N. Slobodetskii) spaces $W_p^{(r)}$ of S. L. Sobolev.

** $K_{\frac{n-r}{2}}$ is the Macdonald function; $c_{n,r}|x|^{(r-n)/2}K_{\frac{n-r}{2}}(|x|) \equiv G_r(x)$.

For $0 < r < 1$ the assertion stated is part of a theorem of E. Stein ([3]); for integral r it follows, for example, from the arguments of the note ([2]). Now let $r > 1$ be a non-integral number and let f be representable in the form (3). We use

Remark 1. The operation of differentiation maps the space $L_p^r(E_n)$ into $L_p^{r-1}(E_n)$ (and this embedding is continuous).

By the embedding theorems ([2]) and on the basis of this remark, the function f and the derivatives $f^{[r]}$ are estimated through φ in the norm L_p. Moreover, $f^{[r]} \in L_p^{r-[r]}(E_n)$ and, consequently, is representable by a Bessel potential of the form (3)

$$f^{[r]} = \int_{E_n} G_{r-[r]}(x-t)g_{[r]}(t)\,dt, \qquad g_{[r]} \in L_p(E_n),$$

where $\|g_{[r]}\|_{L_p} \le c\|\varphi\|_{L_p}$. Since $0 < r-[r] < 1$, by the above-mentioned Stein theorem,

$$\Lambda_p^r(f^{[r]}, E_n) \le A\|g_{[r]}\|_{L_p}.$$

Finally we have

$$\|f\|_{W_p^{[r]}(E_n)} + \Lambda_p^r(f^{[r]}, E_n) \le c\|\varphi\|_{L_p(E_n)},$$

which proves the right-hand part of inequality (4).

Let us prove the left embedding (4). From the finiteness of $\|f\|_{L_p^r(\Omega)}$ it follows, first, that $f \in W_p^{[r]}(E_n)$, and, second, that $\Lambda_p^r(f^{[r]}, E_n)$ is finite. By Stein' s theorem the latter means that $f^{[r]} \in L_p^{r-[r]}(E_n)$. Thus, we have

$$(1+|\lambda|^2)^{[r]/2}\tilde{f} = \tilde{g}, \qquad \|g\|_{L_p} \le c\|f\|_{W_p^{[r]}(E_n)},$$

$$(1+|\lambda|^2)^{\frac{r-[r]}{2}}\widetilde{f^{[r]}} = (1+|\lambda|^2)^{\frac{r-[r]}{2}}(i\lambda)^{[r]}\tilde{f} = \tilde{g}_{[r]}, \qquad \|g_{[r]}\|_{L_p} \le c\Lambda_p^r(f^{[r]}, E_n).$$

Using the linearity of the Fourier transform and the theorem on multipliers, we obtain from this

$$(1+|\lambda|^2)^{r/2}\tilde{f} = \widetilde{\varphi},$$

where the function φ is represented in the form of a linear combination of functions each of which is estimated through g or $g_{[r]}$ in the norm $L_p(E_n)$, i.e., in the final result, we obtain the assertion completing the proof of the theorem:

$$\|\varphi\|_{L_p(E_n)} \le c[\|f\|_{W_p^{[r]}(E_n)} + \Lambda_p^r(f^{[r]}, E_n)].$$

Here it is appropriate to say that the exceptional status of integral r in the definition of $L_p^r(\Omega)$ could have been avoided by using second differences of f (see ([3])). Theorem 1 shows that the integer-valued (with respect to r) classes $L_p^r(\Omega) \equiv L_p^r(E_n)$ are not exceptional in the functional scale under consideration.

2. We now prove the extension theorem, which in essence justifies considering the spaces $L_p^r(\Omega)$ for $\Omega \ne E_n$ in the definition given above.

Theorem 2. *If the boundary Γ of the domain Ω belongs to the class $C^{[r]+1}$, then any function $f \in L_p^r(\Omega)$ can be extended to the whole space E_n with preservation of the class, i.e., for the extended function F the inequality holds*

$$\|F\|_{L_p^r(E_n)} \le c\|f\|_{L_p^r(\Omega)},$$

where the constant c does not depend on f.

In the case of integral r the theorem was proved in ([4]) (see also ([5])). For non-integral r we shall carry out the calculations in the simplest case, when Ω coincides with the upper half-space E_n^+ $(x_n > 0)$. Define the extended function as follows:

$$F(x) = \begin{cases} f(x), & \text{for } x_n > 0 \ (x \in E_n^+), \\ \sum_{k=1}^{[r]+1} \lambda_k f\left(x_1, \dots, x_{n-1}, -\frac{1}{k} x_n\right), & \text{for } x_n < 0 \ (x \in E_n^-). \end{cases}$$

where the numbers λ_k are chosen from the conditions

$$(-1)^l \lambda_1 + \left(-\frac{1}{2}\right)^l \lambda_2 + \dots + \left(-\frac{1}{[r]+1}\right)^l \lambda_{[r]+1} = 1, \qquad l = 0, \dots, [r].$$

It is clear that the function $F(x) \in W_p^{[r]}(E_n)$ (see ([4,5])), and it remains only to verify the finiteness of the expressions

$$[\Lambda_p^r(F^{[r]}, E_n)]^p \equiv [\Lambda]^p = \int_{E_n} dy \left(\int_{E_n} \frac{|F^{[r]}(x) - F^{[r]}(y)|^2}{|x-y|^{n+2(r-[r])}} \, dx \right)^{\frac{p}{2}}.$$

We have, obviously,

$$[\Lambda]^p \le 2^{\frac{p}{2}-1} \left\{ \int_{E_n^+} dy \left(\int_{E_n^+} \dots dx \right)^{\frac{p}{2}} + \int_{E_n^-} dy \left(\int_{E_n^+} \dots dx \right)^{\frac{p}{2}} \right.$$
$$\left. + \int_{E_n^+} dy \left(\int_{E_n^-} \dots dx \right)^{\frac{p}{2}} + \int_{E_n^-} dy \left(\int_{E_n^-} \dots dx \right)^{\frac{p}{2}} \right\}. \tag{5}$$

The first of the integrals in braces exists by assumption—it is $[\Lambda_p^r(F^{[r]}, E_n^+)]^p$. Let us show, for example, the estimate of the second integral:

$$\int_{E_n^-} dy \left(\int_{E_n^+} \frac{\left| \frac{\partial^{[r]} F(x)}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} - \frac{\partial^{[r]} F(y)}{\partial y_1^{l_1} \dots \partial y_n^{l_n}} \right|^2}{|x-y|^{n+2(r-[r])}} \, dx \right)^{\frac{p}{2}} =$$

$$= \int_{E_n^-} dy \left(\int_{E_n^+} \frac{\left| \sum \lambda_k \left(-\frac{1}{k}\right)^{l_n} \left[f^{[r]}(x_1, \dots, x_n) - f^{[r]}\left(y_1, \dots, y_{n-1}, -\frac{1}{k} y_n\right) \right] \right|^2}{|x-y|^{n+2(r-[r])}} \, dx \right)^{\frac{p}{2}}.$$

We have used the fact that

$$\sum_{k=1}^{[r]+1} \lambda_k \left(-\frac{1}{k}\right)^{l_n} = 1.$$

Applying further Hölder's inequality for sums and making the change of variables $\eta_1 = y_1, \dots, \eta_{n-1} = y_{n-1},\ \eta_n = -\frac{1}{k} y_n$, we obtain

$$\leq ([r]+1)^{\frac{p}{2}} \sum_{k=1}^{[r]+1} |\lambda_k|^p k^{\frac{p}{2} - p l_n} \times$$

$$\times \int_{E_n^+} d\eta \left(\int_{E_n^+} \frac{|f^{[r]}(x_1, \dots, x_n) - f^{[r]}(\eta_1, \dots, \eta_n)|^2 \, dx}{[(x_1 - \eta_1)^2 + \dots + (x_{n-1} - \eta_{n-1})^2 + (x_n + k\eta_n)^2]^{n/2 + r - [r]}} \right)^{\frac{p}{2}} <$$

$$< c \int_{E_n^+} d\eta \left(\int_{E_n^+} \frac{|f^{[r]}(x_1, \dots, x_n) - f^{[r]}(\eta_1, \dots, \eta_n)|^2}{|x - \eta|^{n + 2(r - [r])}} \, dx \right)^{\frac{p}{2}}.$$

The estimates of the remaining integrals on the right in (5) proceed analogously, and the theorem for $\Omega = E_n^+$ may be considered proved.

It follows from the computations given how one can (by resorting to a sufficiently regular transformation of the domain Ω) extend a function $f \in L_p^r(\Omega)$ beyond the domain locally in a neighborhood of each boundary point. By virtue of the results of Whitney and Hestenes, this guarantees the extendability of f to the whole space E_n. The theorem is proved.

Theorem 2 makes it possible to carry over the results of Note ([2]) to the case of a domain Ω with sufficiently smooth boundary; namely, the following is valid:

Theorem 3. The function $f(x) \in L_p^r(\Omega)$ on any sufficiently smooth manifold $S \subset \overline{\Omega}$ of dimension m $(1 \leq m \leq n)$ belongs to the space $L_{p'}^{r'}(S)$, where the parameters r, p, n, r', p', m are related by

$$2 \leq p < p' < \infty, \qquad r - \frac{n}{p} \geq r' - \frac{m}{p'}, \qquad 0 \leq r' < r$$

(if $m = n$, the equality $p = p'$ is also allowed). The indicated embedding

$$L_p^r(\Omega) \subset L_{p'}^{r'}(S)$$

is continuous.

Remark 2. The space $L_{p'}^{r'}(S)$ for $m \neq n$, in the case where the manifold S is not flat, is understood as follows. It is assumed that the manifold S admits a finite covering by "simple" manifolds, each of which is sufficiently smoothly mapped onto a domain of the Euclidean space E_m. If the function $f(x)$, considered on the manifold S, under such a mapping is a function of class $L_{p'}^{r'}$ in the Cartesian image of each element of the covering, then we say that

$$f(x)\big|_S \in L_{p'}^{r'}(S).$$

However, one may also use the invariant notation of the integral $\Lambda_p^r(f^{[r]}, S)$, without resorting to a covering of S.

Remark 3. Since the derivatives of f normal to S (for $m \neq n$) are expressed in terms of partial derivatives, and for the latter Remark 1 applies, it follows from Theorem 3 and this remark that we also obtain statements concerning normal derivatives.

3. In conclusion we shall give statements on the properties of functions from $L_p^r(E_n)$ for $rp > n$ $\left(1 < p < \infty,\ \alpha = r - \frac{n}{p} - \left[r - \frac{n}{p}\right]\right)$.

Theorem 4. From $f(x) \in L_p^r(E_n)$ it follows that

$$\Delta_1\left(f^{\left[r-\frac{n}{p}\right]}, h\right) = \left|f^{\left[r-\frac{n}{p}\right]}(x+h) - f^{\left[r-\frac{n}{p}\right]}(x)\right| = o\left(|h|^\alpha\right) \quad \text{for noninteger } r - \frac{n}{p},$$

$$\Delta_2\left(f^{r-\frac{n}{p}-1}, h\right) = \left|f^{r-\frac{n}{p}-1}(x+h) - 2f^{r-\frac{n}{p}-1}(x) + f^{r-\frac{n}{p}-1}(x-h)\right| = o(|h|)$$

$$\text{for integer } r - \frac{n}{p} \geq 1.$$

Some further information about the quality of the function $o(t)$ in Theorem 4 is given by

Theorem 5. From $f(x) \in L_p^r(E_n)$ it follows that

$$\int_{E_n} \frac{\left|\Delta_1\left(f^{\left[r-\frac{n}{p}\right]}, h\right)\right|^p}{|h|^{n+p\alpha}}\, dh \leq c\|f\|_{L_p^r(E_n)}, \quad \text{for noninteger } r - \frac{n}{p},$$

$$\int_{E_n} \frac{\left|\Delta_2\left(f^{r-\frac{n}{p}-1}, h\right)\right|^p}{|h|^{n+p}}\, dh \leq c\|f\|_{L_p^r(E_n)}, \quad \text{for integer } r - \frac{n}{p} \geq 1.$$

For integer $r = l$, and also for $f \in L_2^r(E_n)$, Theorem 4, for integer r and $2 \leq p < \infty$, and Theorem 5 follow from embeddings obtained in the work ([6]). Our proofs are based on the representation of f by a Bessel potential and on various estimates of the kernel $G_r(x)$.

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Note: Figure translations are in progress. See original paper for figures.