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Abstract

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On the Operator of Analytic Continuation of Generalized Functions

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The present paper has as its aim to extend the Cauchy integral operator

$$K\rho = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(x')}{x'-z}\,dx', \qquad \operatorname{Im} z \neq 0, \tag{1}$$

to the class of generalized functions. In connection with this we introduce for consideration certain spaces of generalized and analytic generalized functions that are of particular interest to us.

Let $\mathcal{D}$ denote the space of real finite and infinitely differentiable functions $\psi(x)$. One says that a sequence $\{\psi_n\}$ of functions $\psi_n \in \mathcal{D}$ tends to zero in $\mathcal{D}$ if: 1) there exists a bounded interval Δ of the x-axis outside which all the functions ψ_n vanish; 2) the functions ψ_n and their derivatives of arbitrary order tend to zero uniformly for each arbitrarily fixed order of derivatives. Convergence in $\mathcal{D}$ will be denoted by $\psi_n \Rightarrow \psi$ in $\mathcal{D}$. Let (ρ, ψ) be the number obtained by applying the functional ρ to the function $\psi \in \mathcal{D}$. Such functionals are called generalized functions. (For details on generalized functions see ([1]).)

Let us define the notion of an analytic generalized function. Let $\mathcal{R}$ be the space of holomorphic functions $\Phi(z)$ defined in the strip

$$t_a = \{z \mid |\operatorname{Im} z| < a\}, \qquad \text{where } 0 < a \leqslant \infty, \tag{2}$$

and satisfying the inequality

$$\left|\Phi^{(k)}(z)\right| \leqslant \frac{C^{(k)}}{|z|^{k+1}} \quad \text{for } |z| \to \infty, \tag{3}$$

where $C^{(k)}$ is some constant number, $k = 0, 1, 2 \ldots$.

We introduce the notion of convergence in $\mathcal{R}$. One says that a sequence $\{\Phi_n(z)\}$ of functions $\Phi(z) \in \mathcal{R}$ converges to zero in $\mathcal{R}$ if

$$\left|z^{k+1}\Phi_n^{(k)}(z)\right| \leqslant C_n^{(k)}, \tag{4}$$

where $C_n^{(k)} \to 0$ as $n \to \infty$ and for arbitrarily fixed values $k = 0, 1, 2, ...$ in every bounded domain of the strip t_a; here uniform convergence of $z^{k+1}\Phi_n^{(k)}(z)$ to zero as $n \to \infty$ with respect to z is required. Functionals in $\mathcal{R}$, which we denote by $\langle \rho(z), \Phi \rangle$, always have integral representations

$$\langle \rho(z), \Phi \rangle = \int_C \rho(z)\Phi(z)\, dz \tag{5}$$

for all $\Phi(z) \in \mathcal{R}$, where C is a certain contour in the z-plane. The set of linear continuous functionals in $\mathcal{R}$ will be denoted by $\mathcal{R}'$. Such func-

we shall call functionals **analytic generalized functions**.

Finally, we introduce for consideration the space $\mathcal{L}$ of functions $\Phi(x)$, defined on the whole x-axis, infinitely differentiable, satisfying the inequality

$$\left|\Phi^{(k)}(x)\right| \leq \frac{C^{(k)}}{|x|^{k+1}} \quad \text{as } |x| \to \infty, \tag{6}$$

where $C^{(k)}$ is some constant number, $k = 0, 1, 2, ...$, and representable in the form

$$\Phi(x) = \overline{K}\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\Phi(z)}{x - z}\, dz \tag{7}$$

for any $\Phi(z) \in \mathcal{R}$, where L is a contour consisting of two straight lines parallel to the x-axis: one, situated in $\operatorname{Im} z > 0$, is directed in the positive direction of the x-axis, and the other, situated in $\operatorname{Im} z < 0$, is directed opposite to the positive direction of the x-axis.

The notion of convergence in $\mathcal{L}$ is analogous to convergence in $\mathcal{R}$.

Let $\mathcal{L}'$ be the set of linear continuous functionals in $\mathcal{L}$. We note that the space $\mathcal{D} \subset \mathcal{L}$, and the space $\mathcal{D}' \subset \mathcal{L}'$. Therefore not every generalized function from $\mathcal{D}'$ can belong to $\mathcal{L}'$.

Theorem 1. *In order that any generalized function $\rho \in \mathcal{D}'$ belong simultaneously to the space $\mathcal{L}'$, it is sufficient that it satisfy the inequality*

$$|(\rho, \psi(x + x_0))| \leq \frac{C(\psi)}{|x_0|^\alpha} \quad \text{as } |x_0| \to \infty \tag{8}$$

for any $\psi \in \mathcal{D}$, where $C(\psi)$ is some constant number depending on $\psi \in \mathcal{D}$; x_0 is an arbitrary fixed point of the x-axis, $0 < \alpha < 1$.

Theorem 2. *The operator K, conjugate to the operator $\overline{K}$, is defined on the linear manifold of any generalized functions $\rho \in \mathcal{D}'$ (and also $\rho \in \mathcal{L}'$) satisfying inequality (8) for any $\psi \in \mathcal{D}$ (and also $\psi \in \mathcal{L}$).*

We indicate the scheme of proof of this theorem. Construct an averaging kernel $\omega(x-y;h)$, which is a finite and infinitely differentiable function, i.e. $\omega(x-y;h) \in \mathcal{D}$, and which has the property:

$$\frac{1}{\varkappa h}\int_{-h}^{h}\omega(t;h)\,dt = 1, \quad \text{where } \varkappa = \int_{-1}^{+1}\omega(t;1)\,dt.$$

Then the function

$$\rho_h(x) = \frac{1}{\varkappa h}\,(\rho, \omega(x-y;h))$$

will be infinitely differentiable and will satisfy the inequality

$$|\rho_h(x)| \le \frac{C_h}{|x|^\alpha} \quad \text{as } |x| \to \infty,$$

where C_h is some constant number; $0 < \alpha < 1$.

Construct a sequence $\{\rho_h\}$ of generalized functions ρ_h in $\mathcal{D}$ by the formula

$$(\rho_h, \psi) = \int_{-\infty}^{+\infty}\rho_h(x)\psi(x)\,dx \tag{9}$$

for any $\psi \in \mathcal{D}$. On the basis of Theorem 1, the functional ρ_h, satisfying condition (8), can be extended to the whole space $\mathcal{L}$. Consequently, we obtain:

$$\langle\breve{\rho}(z), \Phi\rangle = (\rho, \overline{K}\Phi) \quad \text{for any } \Phi \in \mathcal{R}. \tag{10}$$

The expression $\langle\breve{\rho}(z), \Phi\rangle$ depends linearly and continuously on $\Phi \in \mathcal{R}$. Now, introducing norms in the spaces $\mathcal{L}$ and $\mathcal{R}$, respectively, by the formulas

$$\|\Phi\|_{(-\infty,\infty)} = \max_{\substack{(-\infty,\infty)\\ k=0,1,2,\dots}} \left|x^{k+1}\Phi^{(k)}(x)\right| \quad \text{for any } \Phi \in \mathcal{L};$$

$$\|\Phi\|_G = \max_{\substack{G\\ k=0,1,2,\dots}} \left|z^{k+1}\Phi^{(k)}(z)\right| \quad \text{for any } \Phi \in \mathcal{R},$$

we shall prove the linear and continuous dependence of $\langle\breve{\rho}(z), \Phi\rangle$ on the given generalized function ρ. Thus, equality (10) defines, on the given manifold of our generalized functions, a **linear operator** of the form:

$$\breve{\rho}(z) = K\rho = \frac{1}{2\pi i}\left(\rho, \frac{1}{x-z}\right), \quad \operatorname{Im} z \neq 0, \tag{11}$$

where $\dfrac{1}{x-z} \in \mathcal{L}$ for $\operatorname{Im} z \neq 0$.

The operator K transforms manifolds of arbitrary generalized functions from $\mathcal{D}'$ and $\mathcal{L}'$, satisfying inequality (8), into analytic generalized functions defined by equality (11) and tending to zero at infinity.

We shall call the operator K the operator of analytic continuation of generalized functions. The uniqueness of this operator is evident.

On the basis of (10) we have

$$\langle \breve{\rho}^{(k)}(z), \Phi\rangle = (-1)^k \langle \breve{\rho}(z), \Phi^{(k)}\rangle =$$

$$= \left(\rho, \frac{k!}{2\pi i}\int_L \frac{\Phi(z)}{(x-z)^{k+1}}\,dz\right) \quad \text{for any } \Phi \in \mathcal{R}. \tag{12}$$

This equality defines the derivative of order k of the analytic generalized function $\breve{\rho}(z)$ defined by formula (11). Thus,

$$\breve{\rho}^{(k)}(z) = \frac{k!}{2\pi i}\left(\rho, \frac{1}{(x-z)^{k+1}}\right), \quad \operatorname{Im} z \neq 0, \quad k = 0, 1, 2, ... \tag{13}$$

Theorem 3. *If the conditions of Theorem 2 are satisfied, the formula*

$$\rho_+ - \rho_- = \rho, \tag{14}$$

holds, where

$$\breve{\rho}_\pm = -\frac{1}{2\pi i}\left(\rho * P\frac{1}{x}\right) \pm \frac{1}{2}\rho. \tag{15}$$

Here the convolution $\left(\rho * P\dfrac{1}{x}\right)$ exists and is defined by the formula

$$\left(\rho * P\frac{1}{x}, \varphi\right) = \left(\rho, \int_0^\infty \frac{1}{y}\left[\varphi(x+y) - \varphi(x-y)\right] dy\right) \tag{16}$$

for any $\varphi \in \mathcal{L}$. The generalized functions $\breve{\rho}_\pm$ mean:

$$(\breve{\rho}_\pm, \varphi) = \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} \breve{\rho}(x + i\varepsilon)\varphi(x)\,dx \tag{17}$$

for any $\varphi \in \mathcal{L}$.

Let us give several simplest examples.

1. $\rho = \delta^{(k)}(x)$, $k = 0, 1, 2, \ldots$. By definition $(\delta^{(k)}(x), \varphi) = (-1)^k \varphi^{(k)}(0)$ for any $\varphi \in \mathcal{L}$. Then, according to formula (11), we obtain

which gives the corresponding analytic generalized function

$$\breve{\delta}^k(z) = \frac{(-1)^{k+1} k!}{2\pi i} \frac{1}{z^{k+1}}, \qquad k = 0, 1, 2, \ldots; \tag{18}$$

$$\breve{\delta}^{(k)}_{\pm} = (-1)^{k+1} \frac{k!}{2\pi i} P_f \frac{1}{x^{k+1}} \pm \frac{1}{2} \delta^{(k)}(x); \tag{19}$$

$$\breve{\delta}^{(k)}_{+} - \breve{\delta}^{(k)}_{-} = \delta^{(k)}(x), \qquad k = 0, 1, 2, \ldots. \tag{20}$$

2. $\rho = P_f \dfrac{1}{x^m}$, where m is a positive integer. By definition

$$\left(P_f \frac{1}{x^m}, \varphi\right) = \int_0^\infty \frac{1}{x^m} \left[\varphi(x) + (-1)^m \varphi(-x) - 2 \sum_{k=0}^{m-1} \frac{x^k}{k!} \varphi^{(k)}(0)\right] dx \tag{21}$$

for every $\varphi \in \mathcal{L}$.

Using this formula, from (11) we obtain the corresponding analytic generalized function, namely:

$$\breve{\rho}(z) = \begin{cases} \dfrac{1}{2z^m}, & \operatorname{Im} z > 0, \\ -\dfrac{1}{2z^m}, & \operatorname{Im} z < 0; \end{cases} \tag{22}$$

$$\breve{\rho}_{\pm} = \pm \frac{1}{2} P_f \frac{1}{x^m} + \frac{i\pi}{2} \frac{(-1)^{m-1}}{(m-1)!} \delta^{(m-1)}(x), \tag{23}$$

whence

$$\breve{\rho}_{+} - \breve{\rho}_{-} = P_f \frac{1}{x^m} \tag{24}$$

and so on.

It should be noted that in their recent work Bremermann and Durand [(2)] made an attempt to define the notions of the Cauchy integral for generalized functions, with the aim of defining a generalized function as the weak jump of two holomorphic functions on the real axis.

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2. H. J. Bremermann, L. Durand, *J. Math. Phys.*, **2**, No. 2 (1961).

Note: Figure translations are in progress. See original paper for figures.