# MATHEMATICS

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## Abstract

## Full Text

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# A STRENGTHENING OF THE UNIQUENESS THEOREM FOR AN $O$-CURVE IN $N_2$

*(Presented by Academician V. I. Smirnov on 2 IV 1962)*

Consider the equation

$$\alpha(r)\frac{d\varphi}{dr} = \Phi(r,\varphi) + \psi(r,\varphi) \equiv \Psi(r,\varphi) \tag{1}$$

under the following assumption.

**Condition A.** 1) The function $\alpha(r)$ is continuous and positive on the interval $(0, r_1]$, $r_1 > 0$ is constant,

$$\int_0^{r_1} \frac{dr}{\alpha(r)} = +\infty;$$

2) the functions $\Phi(r,\varphi)$ and $\psi(r,\varphi)$ are continuous in the domain ($\varphi_1 > 0$ is constant)

$$0 < r \leq r_1, \qquad -\varphi_1 \leq \varphi \leq \varphi_1; \tag{2}$$

$\Phi(r,0) \equiv 0$; for $\varphi \neq 0$, $\varphi\Phi(r,\varphi) < 0$; $|\Phi(r,\varphi)| \geq a|\varphi|^k$, $a > 0$, $k > 0$ are constants; $\psi(r,\varphi) \to 0$ as $r \to 0$, uniformly with respect to $\varphi \in [-\varphi_1, \varphi_1]$.

Under this condition the domain

$$0 < r \leq r_0, \qquad -\varphi_0 \leq \varphi \leq \varphi_0, \tag{3}$$

if the constants $r_0 > 0$ and $\varphi_0 > 0$ are sufficiently small, will be a normal Frommer domain of the second type (an $N_2$-domain) for equation (1), and for it there arises the problem of uniqueness of the $O$-curve (integral curve of equation (1) adjoining the point $r = 0, \varphi = 0$ from this domain).

In note ([1]) a uniqueness theorem for an $O$-curve in $N_2$ was proved for an equation close to (1). The following theorem contains much less restrictive uniqueness conditions (including for the equation considered in (1)).

**Theorem.** *Suppose that for equation (1):*

1) *condition A is satisfied;*

2)

$$\frac{\psi(r,\varphi)}{\omega^{\sigma}(r)} \to 0$$

*as* $r \to 0$*, uniformly with respect to* $\varphi \in [-\varphi_0, \varphi_0]$*, where* $\omega(r)$ *is a fixed function of class* $C^1$ *on* $(0, r_0]$*;* $\omega(r) > 0$*,* $\omega'(r) > 0$ *for* $r \in [0, r_0]$*;* $\omega(r) \to 0$ *as* $r \to 0$*;* $\sigma > 0$ *is a fixed number;*

3) *there exists a number* $u_0 > 0$ *such that in the subdomain* $|\varphi| \leq u_0 \omega^{\sigma/k}(r)$ *of the domain (3), for* $\bar{\varphi} > \bar{\bar{\varphi}}$*,*

$$\Psi(r,\bar{\varphi}) - \Psi(r,\bar{\bar{\varphi}}) \leq \frac{\sigma}{k}\Lambda(r)(\bar{\varphi} - \bar{\bar{\varphi}}),$$

*where* $\Lambda(r)$ *is continuous on* $(0, r_0]$ *and satisfies the inequality (*$M$ *is constant,* $r$ *is any point of* $(0, r_0]$*),*

$$\int_r^{r_0} \left( \frac{\Lambda(r)}{\alpha(r)} - \frac{\omega'(r)}{\omega(r)} \right) dr \leq M < +\infty.$$

*Then equation (1) has a unique* $O$*-curve in the domain (3).*

**Proof.** Analogously to how this was done in ([1]), it is easy to show that the problem under consideration is equivalent to the problem of uniqueness of the $O$-curve of the equation

$$\frac{\omega(r)}{\omega'(r)} u' = \frac{\Psi\left(r, u\,\omega^{\sigma/k}(r)\right)}{\alpha(r)\omega^{\sigma/k-1}(r)\omega'(r)} - \frac{\sigma}{k} u \equiv U(r,u) \tag{4}$$

in the domain

$$0 < r \leq r_0, \qquad -u_0 \leq u \leq u_0. \tag{5}$$

Let $u = u_1(r)$ be an $O$-curve of equation (4) from the domain (5). Putting $u = u_1(r) + v$ in (4), we obtain

$$\frac{\omega(r)}{\omega'(r)} v' = U(r, u_1(r) + v) - U(r, u_1(r)). \tag{6}$$

We shall show that this equation has no solutions $v(r)$ possessing the property $v(r) \geq 0$ for $0 < r \leq r_0$, $v(r) \not\equiv 0$, $v(r) \to 0$ as $r \to 0$. This, evidently, will prove the theorem.

According to condition 3) of the theorem, in the domain

$$0 < r \leq r_0, \qquad 0 < v \leq u_0 - u_1(r) \tag{7}$$

the inequality

$$U(r, u_1(r) + v) - U(r, u_1(r)) \leq \frac{\sigma}{k}\left(\frac{\Lambda(r)\omega(r)}{\alpha(r)\omega'(r)} - 1\right) v$$

is satisfied, i.e., for any solution of equation (6),

$$v' \leq \frac{\sigma}{k}\left(\frac{\Lambda(r)}{\alpha(r)} - \frac{\omega'(r)}{\omega(r)}\right) v.$$

Consequently, any solution $v(r)$ of equation (6) with initial data $(r^*, v^*)$ from the domain (7) satisfies the inequality

$$v(r) \geq v^* \exp\left[-\frac{\sigma}{k}\int_r^{r^*}\left(\frac{\Lambda(r)}{\alpha(r)} - \frac{\omega'(r)}{\omega(r)}\right) dr\right], \qquad r \in (0, r^*],$$

i.e., $v(r)$ does not tend to zero as $r \to 0$. The theorem is proved.

It is easy to verify that the present theorem includes, as special cases, many uniqueness theorems for an $O$-curve in $N_2$ obtained earlier by other authors (the results of Lonn, Vinograd and Grobman, Kukles, Gruz ([2–6])).

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## CITED LITERATURE

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* In paper (1) I allowed an inaccuracy: the Lonn lemma ([2]) follows from the theorem of that paper only in the case when, in the conditions of the latter, the coefficient of Lipschitz $\lambda(r)$ may be considered comparable from below with

$\omega(r)$, $r$. The theorem of the present article includes Lonn' s lemma completely (as a very special case).

*Note: Figure translations are in progress. See original paper for figures.*