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Abstract

Full Text

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ON THE NUMERICAL SOLUTION OF THE DIRICHLET PROBLEM FOR POISSON EQUATIONS

(Presented by Academician S. L. Sobolev, 24 II 1962)

The most perfect error estimates for the numerical solution of the Dirichlet problem for the Laplace equation are contained in the works ^(1,2). In these works, estimates of the growth of derivatives of harmonic functions as one approaches the boundary of the domain are used essentially. In particular, in ⁽²⁾ it is shown that if the harmonic function sought $u \in H(P, A, \lambda)$ ⁽³⁾, then the error $\varepsilon(\Delta^h, P, A, \lambda)$ of the simplest difference analogue of the Dirichlet problem on a uniform grid with linear interpolation at the boundary nodes will be $O(Ah^{\min(P+\lambda, 2)} |\log h|^{\delta_{P+\lambda, 2}})$, where δ_{ij} is the Kronecker symbol. For the Poisson equation analogous estimates are lacking. In ⁽¹⁾, for example, it is proposed to split the problem conditionally into two parts—into the solution of the Laplace equation with a function prescribed on the boundary and into the solution of the Poisson equation with zero boundary conditions, and for the latter problem to use the Gershgorin or Collatz estimates $O(h)$ or $O(h^2)$, depending on whether the exact solution of the Poisson equation with zero boundary conditions has third or fourth bounded derivatives. It follows from this that one must assume a certain smoothness of the solutions of both problems, which seems somewhat artificial. In the theorem proved below we require smoothness of the sought solution and of the right-hand side of the Poisson equation (and not of the solution of the Poisson equation with the prescribed right-hand side).

Theorem. If, instead of the boundary-value problem

$$\begin{aligned} \Delta u &= f(M) \quad \text{in } G, \\ u &= \psi(s) \quad \text{on } S, \end{aligned} \tag{1}$$

where $f \in H(P, A, \lambda)$ in the domain $G' \supset G$, the minimal distance from the boundary of which to Γ is $\varepsilon > 0$, and $u \in H(P_1, A_1, \lambda_1)$ in G , one solves its difference analogue

$$L_\alpha^h u_\alpha = Ch^2 f_\alpha + \sum_{\gamma=1}^k h^{m_\gamma} G_{\gamma, \alpha}, \tag{2}$$

where C is a constant; $G_{\gamma, \alpha}$ are functionals expressed in terms of f, ψ and their derivatives; L_α^h is a difference operator which, for a sufficiently smooth function,

gives a remainder term $O(h^s)$ and satisfies the conditions of the basic lemma of work (2)*, then for the error $\eta = u - \bar{u}$ at grid points the following estimate holds:

$$|\eta| = \varepsilon(L^h, P_1, A_1, \lambda_1) + O(h^{\min(P+\lambda, s-2)}),$$

where $\varepsilon(L^h, P_1, A_1, \lambda_1)$ is the error in the numerical solution, by means of the operator L^h , of the Dirichlet problem for a harmonic function $u \in H(P_1, A'_1, \lambda_1)$.

Proof. From the Weierstrass theorem on the approximation of continuous functions it follows that one can construct a domain \bar{G} with boundary

* These conditions are satisfied in practice for all difference equations with positive coefficients.

$\bar{S} \in \bar{H}_{p+1}(B, l)$ (3), lying entirely in the domain $G' - G$. We represent the solution of problem (1) in the form $u = u_1 + u_2$, where

$$\begin{aligned} \Delta u_1 &= f(M) && \text{in } \bar{G}, \\ u_1 &= 0 && \text{on } \bar{S}; \end{aligned} \quad (3)$$

$$\begin{aligned} \Delta u_2 &= 0 && \text{in } G, \\ u_2 &= \psi(S) - u_1 && \text{on } S. \end{aligned} \quad (4)$$

From (4) it follows that in the planar and spatial cases, for $\lambda < 1$, $u_1 \in H(P + 2, A_2, \lambda)$. As for the solution of problem (4), since it is the difference of two functions $u \in H(P_1, A_1, \lambda_1)$ and $u_1 \in H(P + 2, A_2, \lambda)$, it must belong to the class $H(P_1, A'_1, \lambda_1)$, since $P_1 \leq P + 2$, and when $P_1 = P + 2$, $\lambda_1 \leq \lambda$. It can be shown that the error satisfies the system

$$L_\alpha^h \eta_\alpha = h^{n_\alpha} M_\alpha(u_1) + h^{m_\alpha} M_\alpha(u_2), \quad (5)$$

where $M_\alpha(u_1)$ and $M_\alpha(u_2)$ are expressed through the partial derivatives of u_1 and u_2 , respectively. We write the solution of system (5) in the form $\eta = \bar{\eta} + \tilde{\eta}$, where

$$L_\alpha^h \bar{\eta} = h^{n_\alpha} M_\alpha(u_1),$$

$$L_\alpha^h \tilde{\eta} = h^{m_\alpha} M_\alpha(u_2).$$

It is clear that $\tilde{\eta} = \varepsilon(L^h, P_1, A'_1, \lambda_1)$. It remains to prove that $\bar{\eta} = O(h^{\min(P+\lambda, s-2)})$. For $s \leq P$ this follows directly from the main lemma (2). In order, when $s > P$, to apply the indicated lemma, in deriving (2) one must use a Taylor expansion with a remainder term of order $(P + 2)$, add and

then subtract $u^{(P+2)}(\alpha)$ with the corresponding coefficient; taking into account that $|u^{(P+2)}(\alpha) - u^{(P+2)}(\alpha + \xi h)| \leq A|\xi h|^\lambda$, where $|\xi| \leq 1$, we obtain for the remainder term $O(h^{P+2+\lambda})$.

Using the estimate $\varepsilon(\Delta^h, P_1, A_1, \lambda_1)$ from (2) for the simplest mesh approximation of the Poisson equation (with linear interpolation at boundary nodes), we obtain the following error estimate for the numerical solution of the Dirichlet problem:

$$\eta = O(h^{\min(P_1+\lambda_1, P+\lambda, 2)} |\log h|^{\delta P_1+\lambda_1, 2\delta r, P_1+\lambda_1}), \quad (6)$$

where $r = \min(P_1 + \lambda_1, P + \lambda)$.

Let us note that the order of estimate (6) with respect to h cannot be increased by any positive number. For $r = P_1 + \lambda_1$ this was proved in (2). For $r = P + \lambda$ the proof is completely analogous. To this end we represent the solution of problem (1) in the form (3)

$$u(M) = \int_{\dot{S}} K(M, s)\psi(s) ds + \int_{\dot{G}} R(M, m)f(m) d\tau,$$

where $R(M, m)$ is the Green's function, and for the second integral we use the theorem proved in the introduction of the paper (2).

It is interesting to note that in the case of sufficiently smooth boundaries, depending on whether $u \in H(P + 2, A, \lambda)$ is a solution of the Poisson or Laplace equation, the corresponding unimprovable estimates will be $O(h^{P+\lambda})$ and $O(h^{P+2+\lambda})$.

In the formulation of the theorem, the condition $f \in H(P, A, \lambda)$ in G' may be replaced by the following: there exists a particular solution u_ξ of the equation $\Delta u = \bar{f}(M)$, where $\bar{f}(M)$ in G coincides with $f(M)$, such that $u_\xi \in H(P_\xi, A_\xi, \lambda_\xi)$ in G , where $P_\xi + \lambda_\xi \geq P_1 + \lambda_1$. Then, in the resulting formulation, the theorem proved is an immediate consequence of the main lemma of paper (2) and of the following proposition:

If in G , $u \in H(P, A, \lambda)$, and there exists at least one particular solution $u_\xi \in H(P_\xi, A_\xi, \lambda_\xi)$ of the equation $\Delta u = \bar{f}(M)$, where $P_\xi > P$, then there exists a constant $C(r, P, m)$ such that

$$\left| \frac{\partial^r u}{\partial x_1^{\gamma_1} \dots \partial x_m^{\gamma_m}} \right|_P \leq \frac{C(r, P, m)A}{[\rho(\mathcal{P}, \Gamma)]^{r-P-\lambda}} + A_\xi,$$

where $\sum_{i=1}^m \gamma_i = r$, $P < r \leq P_\xi$, $\rho(\mathcal{P}, \Gamma)$ is the distance from the point \mathcal{P} to Γ .

As an example, consider the problem in a rectangular domain

$$\begin{aligned}\Delta u &= c && \text{in } G, \\ u &= 0 && \text{on } \Gamma,\end{aligned}\tag{7}$$

where $c \neq 0$ is an arbitrary constant. It is known ⁽⁵⁾ that $u \in H(1, A, \lambda)$ for any $\lambda < 1$, and nevertheless estimate (6) for problem (7) will have the form $\eta = O(h^{1+\lambda})$.

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Note: Figure translations are in progress. See original paper for figures.

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