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Abstract

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QUASILINEAR ELLIPTIC SYSTEMS OF EQUATIONS CONTAINING SUBORDINATE TERMS

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1. We first consider systems of equations in divergence form:

$$L(u) \equiv \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h, \qquad x = (x_1, \dots, x_n) \in G, \tag{1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $u = (u_1, \dots, u_N)$; $A_\alpha = (A_\alpha^1, \dots, A_\alpha^N)$ depend, generally speaking, on all derivatives $D^\gamma u$ with $|\gamma| \le m$, $D^0 u \equiv u$ (see the notation in (1)). On the boundary Γ of the domain G, the boundary conditions of the first boundary-value problem are prescribed:

$$u\big|_\Gamma = \varphi_0(x'), \dots, D^\omega u\big|_\Gamma = \varphi_\omega(x'), \qquad x' \in \Gamma, \quad |\omega| \le m-1. \tag{2}$$

The following four conditions are assumed to be satisfied:

1)

$$A(w; v, v) \equiv \sum_{|\alpha|,|\beta|\le m} [A_{\alpha\beta}(x, D^\gamma w) D^\beta v, D^\alpha v] \ge$$

$$\ge \sum_{\delta, j} [\varphi_{\delta j}(D^\delta w_j) D^\delta v_j, D^\delta v_j] \tag{3}$$

$(j = 1, \dots, N)$; $A_{\alpha\beta} = |A_{\alpha\beta}^{lk}|$, $A_{\alpha\beta}^{lk} = \partial A_\alpha^l / \partial D^\beta u_k$; on the right δ runs at least through all the same values as γ with $|\gamma| = m$; $\varphi_{\delta j}(t)$ are, for simplicity, functions of power growth* at infinity: $\varphi_{\delta j}(t) = O(t^{\lambda_j})$ for $t \ge R$; $\lambda_j \ge 0$, λ_j (for brevity of notation) does not depend on δ; $\varphi_{\delta j}(t) \ge c^2 t^s$ for small t; $\varphi_{\delta j}(t) > 0$ for $t \ne 0$ (except for a finite set of values of t); w is any function of the space $W = W_{\bar p}^{(m)}$, $\bar p = (p_1, \dots, p_N)$, $p_j = 2 + \lambda_j$, in which the norm is given by the formula $\|w\|_{m,\bar p} = \|w\|_W = \sum \|w_j\|_{m,p_j}$; $v \in \overset{\circ}{W}$, i.e. $v \in W$ and on Γ satisfies the homogeneous boundary conditions (2).

2)

$$|A_\alpha^l(x,\xi_\gamma)| \le C\left(\sum_{\gamma,j} |\xi_{\gamma j}|^{p_j} + 1\right)^{1-1/p_l}$$

and analogous estimates hold for $\partial A_\alpha^l/\partial x_i$, $\partial^2 A_\alpha^l/\partial x_i^2$;

$$|A_{\alpha\beta}^{lk}(x,\xi_\gamma)| \le C\left(\sum_{\gamma,j} |\xi_{\gamma j}|^{p_j} + 1\right)^{1-1/p_l-1/p_k}.$$

3)

$$|A(w;v,v')| \le CA(w;v,v) + CA(w;v',v')$$

(this is the condition of subordinacy of the skew-symmetric part of the form $A(w;v,v')$, with respect to v and v', to its symmetric part).

4)

$$\sum_{\alpha,\beta,i}\left|\left[\frac{\partial}{\partial x_i}A_{\alpha\beta}(x,D^\gamma w)D^\beta v, D^\alpha v'\right]\right| \le CA(w;v,v) + CA(w;v',v').$$

* The case of non-power growth of $\varphi_{\delta j}$, apparently, is treated analogously with some modifications.

In a somewhat modified, but essentially equivalent, form, these conditions are given in ([1]) and are called the condition of strong ellipticity (p. 3) of the system (1).

Denote by $W^* = W_p^{(-m)}$ (see ([2], [3])) the space conjugate to $\overset{\circ}{W}$, more precisely, the space of generalized functions when the basic space is $\overset{\circ}{W}$. Let $h \in W^*$. By a solution of the problem (1), (2) in the space W is meant a function $u \in W$ such that

$$\sum_\alpha [A_\alpha(x,D^\gamma u), D^\alpha v] = \langle h,v\rangle, \tag{4}$$

where $v \in \overset{\circ}{W}$, and $\langle h,v\rangle$ is the value of the functional h on the function v.

Theorem 1. *If the system (1)–p. 3 (i.e., the conditions 1)–4) are satisfied), then the problem (1), (2) has, and moreover uniquely, a solution $u \in W$ for every right-hand side $h \in W^*$ and for any boundary conditions (2) admitting* *an extension $f(x)$ belonging to the space W inside the domain G (i.e. $D^\omega f|_\Gamma = \varphi_\omega(x')$, $|\omega| \leqslant m-1$).*

We outline the proof of this theorem.

I. First, for the case of smooth $h(x)$ and extension $f(x)$, we construct, by means of the Galerkin method, as indicated in ([1]), a solution $u(x)$ of the problem (1), (2). We represent the sought $u(x)$ in the form $u(x) = f(x) + z(x)$, and find the k-th approximation $z_k = \sum c_{ik} v_i$ $(i = 1, \dots, k)$ to the function z from the system of equations

$$[L(f+z_k), Bv_r] = [h, Bv_r] \quad (r = 1, \dots, k),$$

where $Bv \equiv Mv - \psi(x)\Delta v$, where Δ is the Laplace operator, M is a sufficiently large constant, $\psi(x) > 0$ for $x \in G$, $D^\gamma \psi|_\Gamma = 0$, $|\gamma| \leqslant 2m-1$. The functions $v_r \in \overset{\circ}{W}$ are chosen so that Bv_r also belong to $\overset{\circ}{W}$ and form in $\overset{\circ}{W}$ a complete system of functions (see Lemma 2 in ([1]))*. It is proved that, as $k \to \infty$, $z_k \to z$ and $u = f + z$ is a solution of the problem (1), (2).

II. Introduce the operator $A(u)$ corresponding to the problem (1), (2), by the formula

$$\sum_\alpha [A_\alpha(x, D^\gamma u), D^\alpha v] = \langle A(u), v\rangle, \quad u \in W, \quad v \in \overset{\circ}{W}, \quad A(u) \in W^*. \quad (5)$$

The existence of the operator $A(u)$ $(A : W \to W^*)$ follows from the fact that, by virtue of 2), for fixed $u \in W$ the left-hand side of (5) is a continuous functional on $\overset{\circ}{W}$. Clearly, $A(u)$ is an extension of the operator $L(u)$: $A(u) = L(u)$ for smooth u. From (4) and (5) it follows that u is a solution of the problem (1), (2) if and only if

$$A(u) = h, \qquad u - f \in \overset{\circ}{W}, \quad (6)$$

where $f(x)$ is an extension of the boundary conditions (2) inside G.

Let $W(f)$ be the hyperplane of all $u \in W$ for which $u - f \in \overset{\circ}{W}$.

Lemma 1. *The operator A maps $W(f)$ one-to-one and bicontinuously onto W^* for any fixed $f \in W$.*

The proof follows from two basic inequalities:

$$\|A(u+z) - A(u)\|_{W^*} \leqslant C \sum_{j=1}^{N} \left(\|u_j\|_{m,p_j}^{p_j} + \|z_j\|_{m,p_j}^{p_j} + 1\right)^q \|z\|_W, \quad (7)$$

$u \in W$, $z \in \overset{\circ}{W}$ (whence the continuity of the operator $A(u)$ on $W(f)$ follows); $q = \max(1 - 1/p_k - 1/p_l)$;

$$\|A(u+z)-A(u)\|_{W^*}\cdot\|z\|_W \geqslant \langle A(u+z)-A(u),z\rangle \geqslant \varphi(\|z\|_W), \qquad (7')$$

* We note that the basis system of exponentials $v_k=\exp i(k,x)$ indicated in ([4]) in the case of periodic boundary conditions should be replaced by the system of sines and cosines, since the exposition in ([2]) is carried out in the real case.

where $\varphi(t)=O(t^\rho)$, $\rho=\min p_j\geq 2$, $t\geq R$; $\varphi(t)=O(t^{s_1+2})$, $s_1=\max(\lambda_j,s)$ for small t; $\varphi(t)/t|_{t=0}=0$. From $(7')$ follows the existence of the inverse operator A^{-1} and its continuity. For smooth f it follows from I that equation (6) has a solution for h ranging over a dense set in W^*. From $(7')$ we conclude that (6) is uniquely solvable for arbitrary $h\in W^*$. Further, by a limiting passage in f, we verify the solvability of (6) for arbitrary $f\in W$ and arbitrary $h\in W^*$. The latter proves Lemma 1 and Theorem 1.

With the aid of a process of closure with respect to the coefficients A_α, one can dispense with the conditions of differentiability of these coefficients with respect to the x' s, retaining only the condition of their continuity with respect to the x' s.

Theorem $1'$**.** *Suppose that for $x\in G'$, $\overline{G}\subset G'$, the following are fulfilled: the algebraic analogue of condition 1), i.e. the condition in which $D^\gamma w$ is replaced by the vector ζ_γ, $D^\gamma v-\eta_\alpha$, the integrals [,] by the integrand expressions; that part of condition 2) which pertains to the growth estimates of A^l_α and $A^{lk}_{\alpha\beta}$; the algebraic analogue of condition 3) and the condition*

$$\max_{x\in G'}\sum_{\alpha,\beta}(A_{\alpha\beta}(x,\zeta_\gamma)\eta_\beta,\eta_\alpha)\leq C\sum_{\alpha,\beta}(A_{\alpha\beta}(x,\zeta_\gamma)\eta_\beta,\eta_\alpha), \qquad x\in G'.$$

Then problem (1), (2), or, what is the same, problem (6), has a, and moreover a unique, solution for arbitrary $h\in W^*$, $f\in W$.

For the proof one introduces the operators $L_\tau(u)$ with coefficients $A_{\alpha,\tau}(x,\xi_\gamma)=J_\tau A_\alpha(x,\xi_\gamma)$, where J_τ is the operator of averaging with respect to the x' s. The operators $L_\tau(u)$ satisfy conditions 1)–4) and uniformly in τ the estimates (7), $(7')$. Letting $\tau\to 0$, we derive the assertion of the theorem.

2. General equations with subordinate terms. Systems of the form (1) should be regarded as systems consisting only of principal terms. We now consider equations of a more general form:

$$\mathfrak{A}(u)\equiv\mathcal{L}u+\mathcal{M}u\equiv\sum_{r=1}^{s}L_r(u)+\sum_{|\beta|\leq m}(-1)^{|\beta|}D^\beta V_\beta(x,D^\delta u)=h, \qquad (8)$$

where $L_r(u)$ are operators of order $2m_r$, having in $D^\gamma u_j$ degree of growth $p_{rj}-1=\lambda_{rj}+1$ and satisfying the conditions of Theorem 1 or $1'$; $m=\max m_r$;

$|\beta| + |\delta| \le 2m - 1$. Analogously to §1 we construct extensions $A(u)$ and $V(u)$ of the operators $\mathcal{L}(u)$ and $\mathcal{M}(u)$:

$$\sum_{\alpha,r} [A_{\alpha,r}(x, D^\gamma u), D^\alpha v] + \sum_\beta [V_\beta(x, D^\gamma u), D^\beta v] = \langle A(u) + V(u), v\rangle, \qquad (9)$$

where $u \in W = W_{\bar{p}_1}^{(m_1)} \cap \cdots \cap W_{\bar{p}_s}^{(m_s)}$, $v \in \overset{\circ}{W}$, $A(u), V(u) \in W^* = (\overset{\circ}{W})^*$. The solution of problem (8), (2) is equivalent to finding a solution u of the equation

$$A(u) + V(u) = h, \qquad u - f \in \overset{\circ}{W}, \qquad (10)$$

where $f(x)$ is a continuation into G of the conditions (2). The operator $\mathcal{M}(u)$ is called subordinate if the corresponding operator $V(u)$ ($V : W \to W^*$) is completely continuous on any hyperplane $W(f)$, where $f \in W$ (or, what is more restrictive, on all of W).

Theorem 2. *Suppose the operators $L_r(u)$ satisfy the conditions of Theorem 1 or $1'$, and the operator $\mathcal{M}(u)$ is subordinate.*

a) If for arbitrary $f \in W$ and $z \in \overset{\circ}{W}$ the inequality

$$\langle A(f + z) - A(f), z\rangle + \langle V(f + z) - V(f), z\rangle \ge \varphi(\|z\|_W), \qquad (11)$$

is fulfilled, where $\|z\|_W = \sum \|z\|_{m_r,\bar{p}_r}$, $\varphi(t) = O(t^\rho)$ for $t \ge R$, $\rho \ge 1$, $\varphi(t)$ is an increasing function, and $\varphi(t)/t|_{t=0} = 0$, then problem (8), (2), or problem (10), has a, and moreover a unique, solution $u \in W$.

b) If (11) is fulfilled only for large $\|z\|_W$, then problem (8), (2) (or (10)) is always solvable, but uniqueness may also fail.

c) If condition (11) is fulfilled only on the sphere $\|z\| = R$ ($R > \|f\|_W$), then problem (8), (2) (or (10)) is solvable for any h and f satisfying the inequality

$$\|h\|_{W^*} + \|A(f)\|_{W^*} + \|V(f)\|_{W^*} < \varphi(R)/R. \qquad (12)$$

The proof of this theorem follows, roughly speaking, from the fact that the degree of covering of zero under the mapping of the sphere $\|z\| \le R$ by the operator $B_t(z) \equiv f + z - A^{-1}(-tV(f+z) + th)$ is equal to 1 for every t, $0 \le t \le 1$.

3. Criteria for the subordination of the operator $\mathcal{M}(u)$

For simplicity of formulation we restrict ourselves to the case of one equation ($N = 1$) of order $2m$ and $s = 1$.

Theorem 3. Let $V_\beta(x, 0) = 0$, $p \ge 2$, and for arbitrary η_δ, ξ_δ

$$|V_\beta(x, \eta_\delta + \xi_\delta) - V_\beta(x, \eta_\delta)| \le C \sum_\delta (|\eta_\delta|^\mu + |\xi_\delta|^\mu + 1)\xi_\delta, \tag{13}$$

where $\mu = \mu(\delta)$, and the exponents μ are chosen according to the following rules: a) for $|\beta| = m$: $1/(\mu+1)q_1 \ge 1/p - (m - |\delta|)/n$, where $q_1 \ge p/(p-1)$ (if $1/p - (m - |\delta|)/n < 0$, then the corresponding term in (13) may be replaced by an arbitrary function $\varphi(\eta_\delta, \xi_\delta)$); b) for $|\beta| < m$: $1/(\mu+1)q_1 \ge 1/p - (m - |\delta|)/n$, where $q_1 \ge q/(q-1)$, $1/q > 1/p - (m - |\beta|)/n$ (in the case $1/p - (m - |\beta|)/n < 0$, $q_1 = 1$). Then the operator $V(u)$ is completely continuous (on all of W), and consequently the operator $\mathcal{M}(u)$ is subordinate.

In proving this theorem we use the embedding theorems of S. L. Sobolev[5]. We note that the admissible degrees of growth of the coefficients $V_\beta(x, \xi_\delta)$ indicated in Theorem 3 are connected not only with the degrees of growth and the order of the principal terms $A_\alpha(x, \xi_\gamma)$, but also with the fact that $h \in W^*$ and, consequently, may have singularities. In the case of smooth right-hand sides and smooth solutions these orders may be different, as was first established by S. N. Bernstein[6] for elliptic equations of second order (see also[7]).

From Theorems 2 and 3 there follows a number of corollaries on the solvability of problem (8), (2) or problem (10).

Theorem 4. Let an equation of the form (8) ($N = 1$) of order $2m$ be given, where $L(u)$ is a s. e. operator ($s = 1$).

a) If the functions $A_\alpha(x, \xi_\gamma)$ are polynomials in ξ_γ of order $\le 2l-1$, for which condition (3) is fulfilled with $\varphi_{\delta j}(\xi_\delta) = \varphi_\delta(\xi_\delta) = c|\xi_\delta|^{2l-2}$ for $|\xi_\delta| \ge R$, and the functions $V_\beta(x, \xi_\delta)$ are arbitrary polynomials in ξ_δ of order $\le 2l - 2$, then problem (8), (2) (or (10)) is always solvable (for any $h \in W_p^{(-m)}$ and $f \in W_p^{(m)}$, $p = 2l$).

b) If the functions $A_\alpha(x, \xi_\gamma)$ have growth order $p - 1$ in ξ_γ, and $V_\beta(x, \xi_\delta)$ have growth order $p - 1 - \varepsilon$ in ξ_δ (in the sense of fulfillment of (13) with $\mu = p - 2 - \varepsilon$), then problem (8), (2) (or (10)) is always solvable.

The proof of these propositions follows from the fact that the conditions of Theorem 2b) and Theorem 3 are fulfilled. We note that in Theorem 4 no restrictions are imposed on the signs of V_β or of their derivatives. To illustrate Theorem 2c) one may give the equation $\mathfrak{A}(u) \equiv \Delta^2 u + \alpha \sum \frac{\partial}{\partial x_i} \left|\frac{\partial^2 u}{\partial x_i^2}\right|^{2-\varepsilon} + \alpha \sum \left(\frac{\partial^2 u}{\partial x_i^2}\right)^2 = h$ $(i = 1, 2)$; $u|_\Gamma = \frac{\partial u}{\partial n}\big|_\Gamma = 0$. Here $L(u) = \Delta^2 u$, $\mathcal{M}(u)$ is equal to the sum of the remaining terms and, as follows from Theorem 3, is a subordinate operator. This problem, by virtue of Theorem 2c), is always solvable if α and $\|h\|_{W^*}$ do not exceed certain quantities.

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Note: Figure translations are in progress. See original paper for figures.