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Abstract

Full Text

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A Generalization of the Mean Value Theorem for Functions of Several Variables

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The proposed theorem was needed in solving a number of questions in the theory of partial differential equations. It may be regarded as a certain analogue of the mean value theorem for functions of n variables, $n \geq 2$.

Let G be a bounded domain of the n-dimensional space of variables $x_1, \dots, x_n$. The measure of the closed domain $\overline{G}$ is less than 1: $\operatorname{mes} \overline{G} < 1$. The domain G is situated in the strip

$$\Pi\{a < x_1 < b;\ -\infty < x_k < \infty,\ k \geq 2\},$$

$a < 0$; $b > 1$. The intersection of the boundary Γ of the domain G with the $(n-1)$-dimensional planes $x_1 = a$ and $x_1 = b$ is nonempty in both cases. Denote them respectively by Γ_a and Γ_b.

We shall say that a set M separates Γ_a from Γ_b in G if any polygonal line $L \subset G \cup \Gamma_a \cup \Gamma_b$, $L \cap \Gamma_a \neq 0$, $L \cap \Gamma_b \neq 0$, intersects M: $L \cap M \neq 0$. In the closed domain $\overline{G}$ a twice continuously differentiable function $f(x) = f(x_1, \dots, x_n)$ is given, $\operatorname{osc} f(x) < 1$.

It is required to prove that

there exists a finite number of smooth surfaces $S_1, \dots, S_k$ such that

$$\bigcup_{i=1}^{k} S_i \text{ separates } \Gamma_a \text{ from } \Gamma_b \text{ in } G \quad \text{and} \quad \sum_{i=1}^{k} \int_{S_i \cap G} \left| \frac{\partial f}{\partial n} \right| ds < 8$$

(n is the normal to the surface).

Proof. First of all note that $|\partial f / \partial n|$ is the length of the projection onto the direction n of the vector $\operatorname{grad} f$. Therefore, if at every point of an $(n-1)$-dimensional surface S the vector n—the normal to the surface S—and the vector $\operatorname{grad} f$ are orthogonal, then

$$\int_S \left| \frac{\partial f}{\partial n} \right| ds = 0.$$

In what follows it will be convenient for us to assume that the function $f(x)$ is given not only in $\overline{G}$, but in a wider domain $\overline{G_\alpha}$. More precisely, let G_α be the α-expansion of the domain G, and let α be so small that $\operatorname{mes} \overline{G_\alpha} < 1$; extend the function $f(x)$ to G_α in a twice continuously differentiable way, preserving the inequality $\operatorname{osc} f < 1$, and so that in some neighborhood of the boundary Γ_a of the domain G_α one has $\operatorname{grad} f \neq 0$.

Consider in $\overline{G_\alpha}$ the field $\operatorname{grad} f$. We shall prove (in a special lemma) that the set of its singular points can be covered by a finite number of balls $\widetilde{\Omega}_i$, $i = 1, \dots, l$, such that for some β, $\alpha > \beta > 0$, the β-expansions of these balls Ω_i, $i = 1, \dots, l$, belong to G_α and

$$\sum_{i=1}^{l} \int_{\omega_i} |\operatorname{grad} f|\, ds < 1,$$

where

$$\omega_i = \overline{\widetilde{\Omega}_i} \setminus \Omega_i.$$

Introduce the following notation:

$$\widetilde{D} = G_\alpha \setminus \bigcup_{i=1}^{k} \widetilde{\Omega}_i;$$

$\widetilde{\Delta}$ is the boundary of $\widetilde{D}$; $\tilde{l}_Q$ is the trajectory of the field $\operatorname{grad} f$ in $\widetilde{D}$ passing through the point $Q \in \widetilde{D}$; D is the set of points $x \in \widetilde{D}$ whose distance from $\widetilde{\Delta}$ is greater than β; Δ is the boundary of D; l_Q is the trajectory of the field $\operatorname{grad} f$ in D passing through the point $Q \in D$.

We now set forth the plan of the proof. We shall call a domain $P \subset D$ a bundle of trajectories if P, together with each of its points Q, contains the entire trajectory l_Q. The part T of the boundary of the domain P consisting of points $Q \notin \Delta$ obviously has the same property. We shall call T a tube of trajectories. Obviously,

$$\int_T \left| \frac{\partial f}{\partial n} \right| ds = 0.$$

We shall call a tube T through if there exists a broken line $L \subset P$ connecting Γ_a and Γ_b. Suppose that we have succeeded in finding a finite number of bundles of trajectories $P_1, \dots, P_m$ satisfying the requirements:

1) $\bigcup_i \overline{P_i} \supset D$; 2) each through tube T_i can be "partitioned" by an $(n-1)$-dimensional plane in such a way that the part of the plane that falls inside T_i (the "partition" π_i) separates Γ_a from Γ_b inside T_i, and moreover

$$\sum_i \int_{\pi_i} |\operatorname{grad} f|\, ds < 6.$$

Then the union of all tubes (including non-through ones), all partitions π_i, and all spheres ω_j, $j = 1, \dots, l$, is the desired set M.

Bundles P_i satisfying requirements 1), 2) can indeed be found. But the proof of this fact is rather long. We shall construct not a finite, but a countable set of bundles P_i, $\overline{P_i} \cap P_j = 0$ for $i \neq j$, satisfying requirements 1), 2). For any $\varepsilon > 0$ one can choose from our countable set of bundles a finite subset

$$P_1, \dots, P_m,$$

such that the points

$$x \in D \setminus \bigcup_{i=1}^{m} \overline{P_i},$$

for which $x_1 = 1/2$, form a set S_0 whose $(n-1)$-dimensional measure is less than ε: $\operatorname{mes}_{n-1} S_0 < \varepsilon$. Let τ_i be the intersection of P_i with the plane $x_1 = 1/2$, $i = 1, \dots, m$. Denote by τ_i^γ the set of points $x \in \tau_i$ whose distance to the boundary of τ_i is greater than $\gamma > 0$, and let γ be so small that

$$\operatorname{mes}_{n-1}(\tau_i \setminus \tau_i^\gamma) < \varepsilon/m.$$

If τ_i^γ consists of infinitely many components, then we choose from it a finite number of components whose union σ_i differs from τ_i^γ in measure by less than ε/m:

$$\operatorname{mes}_{n-1}(\tau_i^\gamma \setminus \sigma_i) < \varepsilon/m.$$

If τ_i^γ consists of a finite number of components, put $\sigma_i = \tau_i^\gamma$. Denote by S the set of points of the plane $x_1 = 1/2$ that do not belong to

$$\bigcup_{i=1}^{m} \sigma_i \cup \bigcup_{j=1}^{l} \omega_j.$$

For

$$\varepsilon = \frac{1}{3 \max_{x \in \overline{D}} |\operatorname{grad} f(x)|},$$

it is obvious that

$$\int_{S \cap G} |\operatorname{grad} f| \, ds < 1.$$

Therefore the sum

$$S \cup \bigcup_{i=1}^{m} T_i \cup \bigcup_{i=1}^{m} \pi_i \cup \bigcup_{j=1}^{l} \omega_j$$

may be taken as M.

Thus, in order to prove the theorem, it suffices to prove the lemma on the exceptional points of the field grad f and to construct a countable set of bundles P_i satisfying requirements 1), 2).

We begin with the lemma. Let N be the set of points $x \in \overline{G_a}$ at which grad $f(x) = 0$. Represent N as the sum $N = N_1 \cup N_2$, where N_1 consists of the points $x \in N$ at which $d^2 f(x) \neq 0$, and N_2 of the points $x \in N$ at which $d^2 f = 0$.

I. It is easy to show that all density points of the set N_1 belong to N_2, i.e. mes $N_1 = 0$. Therefore, for any $\varepsilon > 0$ there is an open set O_1 such that mes $O_1 < \varepsilon$, $N_1 \subset O_1 \subset \overline{G_a}$. For each point $x \in N_1$ construct a ball Ω of radius $r(\Omega)$ so small that $\Omega \subset O_1$. Denote the set of all such balls by M_1. Let

$$R_1 = \sup_{\Omega \in M_1} r(\Omega).$$

Select from M_1 a finite or countable set of balls according to the following rule. As Ω_1 we take an arbitrary ball $\Omega \in M_1$ such that $r(\Omega) > R_1/2$. Suppose that $\Omega_1, \dots, \Omega_{i-1}$ have already been chosen and are pairwise disjoint.

Denote by M_i the set of balls $\Omega \in M_1$ that do not intersect any of the balls $\Omega_1, \dots, \Omega_{i-1}$. If M_i is empty, the selection of balls is completed. Otherwise we take for Ω_i an arbitrary ball $\Omega \in M_i$ such that $r(\Omega) > R_i/2$, where $R_i = \sup_{\Omega \in M_i} r(\Omega)$. The balls Ω_i are pairwise disjoint, so that $e_n \sum_i r^n(\Omega_i) < \varepsilon$, where e_n is the volume of the unit n-dimensional ball.

Next, for each i we construct balls Ω_i' and $\widetilde{\Omega}_i'$, concentric with Ω_i, with radii $r_i' = 10r(\Omega_i)$ and $\tilde{r}_i' = 5r(\Omega_i)$. It is clear that the balls $\widetilde{\Omega}_i'$ cover N_1. Moreover, we may assume that the balls $\Omega \in M$ were chosen so small that $\Omega_i' \subset G_\alpha$. For every i, at the center of the ball Ω_i', grad $f = 0$. Therefore (in view of the boundedness of the second derivatives $f(x)$) on the sphere $\omega_i = \overline{\Omega}_i' - \Omega_i'$, $|\operatorname{grad} f| \leq c r_i'$, where c is a constant independent of i.

Thus,

$$\sum_i \int_{\omega_i'} |\operatorname{grad} f| \, ds \leq \sum_i c r_i' \int_{\omega_i'} ds = \sum_i c r_i' \cdot s_n r_{i'}^{n-1},$$

where s_n is the area of the surface of the unit $(n-1)$-dimensional sphere. Consequently,

$$\sum_i \int_{\omega_i'} |\operatorname{grad} f| \, ds \leq c s_n \sum_i r_{i'}^n < \frac{10^n c s_n}{e_n} \varepsilon < \frac{1}{2} \quad \text{for} \quad \varepsilon = \frac{e_n}{2 \cdot 10^n c s_n}.$$

II. Let $x^0 = (x_1^0, \dots, x_n^0) \in N_2$, i.e. $\operatorname{grad} f(x^0) = d^2 f(x^0) = 0$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that at every point $x = (x_1, \dots, x_n) \in G_\alpha$,

$$\sum_{i=1}^{n} (x_i - x_i^0) = r^2 \leq \delta,$$

the inequality $|\operatorname{grad} f(x)| \leq \varepsilon r$ holds, and hence

$$\int_{\sum_{i=1}^n (x_i - x_i^0)^2 = r^2} |\operatorname{grad} f|\, ds \leq \varepsilon s_n r^n.$$

Around each point $x^0 \in N_2$ describe a ball $\Omega \subset G_\alpha$ of radius r such that

$$\int_{\sum_{i=1}^n (x_i - x_i^0)^2 = r^2} |\operatorname{grad} f|\, ds \leq \varepsilon s_n r^n.$$

Let Ω^0 be the ball concentric with Ω of radius $r^0 = r/6$. From the set of balls Ω^0 choose a finite number of balls $\Omega_1^0, \dots, \Omega_s^0$ covering N_2. Further, from this set of balls $\Omega_1^0, \dots, \Omega_s^0$, select a subset $\Omega_1, \dots, \Omega_j$ according to the following rule: for Ω_1 take the largest of the balls Ω_i^0; for Ω_2, the largest of the balls Ω_i^0 not intersecting Ω_1, and so on. The balls $\Omega_1, \dots, \Omega_j$ are pairwise disjoint, so that

$$\sum_{i=1}^{j} \operatorname{mes} \Omega_i < \operatorname{mes} G_\alpha < 1.$$

For each i, $1 \leq i \leq j$, consider the balls Ω_i'' and $\widetilde{\Omega}_i''$, concentric with Ω_i, with radii $r_i'' = 6r(\Omega_i)$ and $\tilde{r}_i'' = 3r(\Omega_i)$. It is clear that

$$\bigcup_{i=1}^{j} \widetilde{\Omega}_i'' \supset N_2, \qquad \sum_{i=1}^{j} \operatorname{mes} \Omega_i'' < 6^n,$$

so that, denoting $\omega_i'' = \overline{\Omega}_i'' \setminus \Omega_i''$, we find

$$\sum_{i=1}^{j} \int_{\omega_i''} |\operatorname{grad} f|\, ds < \varepsilon s_n \sum_{i=1}^{j} r_i^n < \varepsilon \frac{s_n}{e_n} 6^n < \frac{1}{2} \quad \text{for} \quad \varepsilon = \frac{1}{2} \frac{e_n}{s_n 6^n}.$$

Combining the results of parts I and II, we have constructed a finite or countable set of balls

$$\widetilde{\Omega}_1', \dots, \widetilde{\Omega}_i', \dots, \widetilde{\Omega}_1'', \dots, \widetilde{\Omega}_j'',$$

covering N, where the balls

$$\Omega_1', \dots, \Omega_i', \dots, \Omega_1'', \dots, \Omega_j''$$

(concentric with them and of twice the radii) belong to G_α and satisfy the condition

$$\sum_i \int_{\omega_i} |\operatorname{grad} f|\, ds + \sum_{i=1}^{l} \int_{\omega_i} |\operatorname{grad} f|\, ds < 1.$$

The assertion of the lemma follows from this by virtue of the closedness of N.

Let us now pass to the construction of the pencils P_i. Choose $\delta > 0$ so that the following conditions are satisfied:

1°. $\delta < \frac{1}{2}\min(-a,\, b-1)$.

2°. $\delta < 1/\max\limits_{Q\in\widetilde{D}} k(Q)$, where $k(Q)$ is the curvature of the trajectory $\tilde{l}_Q$ at the point Q.

3°. For any two points A and $B \in \widetilde{D}$, the distance between which is less than δ, the inequality

$$|\operatorname{grad} f(A) - \operatorname{grad} f(B)| < 1$$

holds.

Let Ω_Q^ε be the open n-dimensional ball of radius ε with center at the point Q, and let l_Q^δ be the δ-neighborhood of the trajectory $\tilde{l}_Q$. For each point $A \in \overline{D}$ construct the ball Ω_A^ε. Let $\varepsilon = \varepsilon(A) > 0$ be so small that $\Omega_A^\varepsilon \subset \widetilde{D}$, and for any point $Q \in \Omega_A^\varepsilon$, $l_Q \subset l_A^\delta$. From the set of balls Ω_A^ε, $A \in \overline{D}$, choose a finite number of balls covering $\overline{D}$: $\Omega^1, \dots, \Omega^m$.

Let C be an arbitrary component of the intersection $\Omega^i \cap D$. The trajectories l_Q, $Q \in C$, occupy a domain B. When the different domains B intersect, there is formed no more than a countable set of pairwise nonintersecting pencils of trajectories P_i. They obviously satisfy requirement 1). We shall show that requirement 2) is also fulfilled for them.

By construction each pencil P_i belongs to the δ-neighborhood l^δ of some trajectory $\tilde{l} = l_i$. If T_i is a through tube, then by the choice of δ (condition 1°) the trajectory l_i intersects the planes $x_1 = 0$ and $x_1 = 1$. Let l_i' be the arc of l_i enclosed between these planes. Consider

$$\int_{l_i'} |\operatorname{grad} f|\, dl.$$

Since $\operatorname{osc} f < 1$, it follows that

$$\int_{l_i'} |\operatorname{grad} f|\, dl < 1.$$

Therefore the linear measure of the set l_i^2 of points $x \in l_i'$ where $|\operatorname{grad} f(x)| \le 2$ is not less than $1/2$:

$$\operatorname{mes}_1 l_i^2 \ge \frac{1}{2}.$$

Through each point $x \in l_i^2$ draw the $(n-1)$-dimensional plane π_x orthogonal to l_i at the point x. The component of the intersection $\pi_x \cap P_i$ containing the point x will be denoted by π_{xi}. By the choice of δ (condition 2°), $\pi_{xi} \cap \pi_{yi} = 0$ for $x \neq y$. Each section π_{xi} separates Γ_a from Γ_b inside T_i; as π_i take that one of them whose $(n-1)$-dimensional measure is minimal:

$$\operatorname{mes}_{n-1} \pi_i = \min_{x \in l_i^2} \operatorname{mes}_{n-1} \pi_{xi}.$$

We shall prove that

$$\sum_i \int_{\pi_i} |\operatorname{grad} f|\, ds \leq 6$$

(the summation is over all i corresponding to through tubes T_i).

Consider the set

$$\bigcup_i \bigcup_{x \in l_i^2} \pi_{xi} = \pi.$$

Since $\pi \subset G_a$, we have $\operatorname{mes} \pi < 1$. On the other hand,

$$\operatorname{mes} \pi = \sum_i \operatorname{mes} \bigcup_{x \in l_i^2} \pi_{xi} \geq \sum_i \frac{1}{2} \operatorname{mes}_{n-1} \pi_i.$$

Consequently,

$$\sum_i \operatorname{mes}_{n-1} \pi_i \leq 2.$$

Further, by the choice of δ (condition 3°), at points $x \in \pi_i$

$$|\operatorname{grad} f(x)| \leq 3.$$

Thus,

$$\sum_i \int_{\pi_i} |\operatorname{grad} f|\, ds \leq 6.$$

The theorem is thereby proved.

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Note: Figure translations are in progress. See original paper for figures.