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Abstract

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THE FLUX VECTOR AND MASS TENSOR OF A RELATIVISTIC IDEAL GAS

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In paper (^{1}) a kinetic equation was established for the distribution function of a relativistic ideal N-component gas in an arbitrary Einstein gravitational field. Here we shall express the flux vector and the mass tensor of the gas in terms of its distribution function and, on the basis of the kinetic equation, establish that the divergences of these two quantities are equal to zero.

The notation used in this paper is borrowed from (^{1}). In particular, $A_i(x, P)$ is the distribution function of the i-th component of the gas, $i = 1, 2, \ldots, N$.

We shall obtain the flux vector of particles of species i by considering the mean number of such particles crossing a hypersurface S in the state space F_i of particles of species i. Let the hypersurface S be represented as a skew product with layer Π_i and base σ, where Π_i is the momentum space of particles of species i, and σ is a hypersurface in the event space. As displacement elements along the hypersurface S we choose the following vectors:

$$\begin{aligned}
d_1 &= \{d_1x^0,\ d_1x^1,\ d_1x^2,\ d_1x^3,\ 0,\ 0,\ 0\},\\
d_2 &= \{d_2x^0,\ d_2x^1,\ d_2x^2,\ d_2x^3,\ 0,\ 0,\ 0\},\\
d_3 &= \{d_3x^0,\ d_3x^1,\ d_3x^2,\ d_3x^3,\ 0,\ 0,\ 0\},\\
d_4 &= \{0,\ 0,\ 0,\ 0,\ dp^1,\ 0,\ 0\},\\
d_5 &= \{0,\ 0,\ 0,\ 0,\ 0,\ dp^2,\ 0\},\\
d_6 &= \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ dp^3\},
\end{aligned} \tag{1}$$

where the first three vectors are determined by the displacement vectors in the event space along the hypersurface σ. According to the general definition (^{1}), the mean number of particles of species i crossing the hypersurface S is equal to

$$\int_S A_i(x, P)\, d\Sigma = \int_\sigma \sqrt{-g} \begin{vmatrix} n_i^0(x) & n_i^1(x) & n_i^2(x) & n_i^3(x) \\ d_1x^0 & d_1x^1 & d_1x^2 & d_1x^3 \\ d_2x^0 & d_2x^1 & d_2x^2 & d_2x^3 \\ d_3x^0 & d_3x^1 & d_3x^2 & d_3x^3 \end{vmatrix}, \tag{2}$$

where

$$n_i^\alpha(x) = \int_{\Pi_i} p^\alpha A_i(x,P)\,dP. \tag{3}$$

This vector in the event space, in accordance with the meaning of formula (2), is the flux vector of particles of species i (more precisely, it is c times smaller than the flux vector*).

In an analogous way we obtain the mass tensor of the gas. Let $\xi(x)$ be a vector field in event space. Define the physical quantity by the scalar product $(\xi(x), P)$. The mean value of this quantity, carried by particles of species i through the hypersurface S, is equal to

$$\int_S (\xi(x), P)\,A_i(x,P)\,d\Sigma = \int_\sigma \sqrt{-g}\begin{vmatrix} \eta_i^0(x) & \eta_i^1(x) & \eta_i^2(x) & \eta_i^3(x) \\ d_1x^0 & d_1x^1 & d_1x^2 & d_1x^3 \\ d_2x^0 & d_2x^1 & d_2x^2 & d_2x^3 \\ d_3x^0 & d_3x^1 & d_3x^2 & d_3x^3 \end{vmatrix}, \tag{4}$$

where the vector $\eta_i^\alpha(x)$ is formed by contracting the vector $\xi_\beta(x)$ with the tensor $T_i^{\alpha\beta}(x)$, equal to

$$T_i^{\alpha\beta}(x) = \int_{\Pi_i} p^\alpha p^\beta A_i(x,P)\,dP. \tag{5}$$

In accordance with the meaning of formula (4), this tensor is the mass tensor of the i-th component of the gas (more precisely, it is c times smaller than the mass tensor*). The mass tensor of the entire gas is represented by the sum

$$T^{\alpha\beta}(x) = \sum_{i=1}^{N} T_i^{\alpha\beta}(x). \tag{6}$$

The symmetry of the tensor $T_i^{\alpha\beta}$ with respect to the indices α, β is evident. The trace of this tensor is

$$T_i = T_{i,\alpha}^{\alpha} = m_i c^2 \int_{\Pi_i} A_i(x,P)\,dP. \tag{7}$$

If the rest mass m_i of particles of species i is equal to zero, then the trace T_i is also equal to zero. Thus, the mass tensor $T^{\alpha\beta}(x)$ is symmetric. Moreover, if

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* This is connected with the fact that we have represented the volume element of the state space in the form $dX \cdot dP$, and not in the form $(c^{-1}dX)\cdot(cdP)$.

the rest masses of all gas particles are equal to zero, then the trace of the mass tensor $T = T^{\alpha}_{\alpha}(x)$ is also equal to zero.

Let us prove that the divergence of the flux vector $n^{\alpha}_{i}(x)$ of particles of species i and the divergence of the mass tensor $T^{\alpha\beta}(x)$ of the entire gas are equal to zero, i.e., that

$$\nabla_{\alpha} n^{\alpha}_{i} = 0, \qquad \nabla_{\alpha} T^{\alpha\beta} = 0. \tag{8}$$

For this purpose consider a closed hypersurface S^{*}, represented as the oblique product with the layer Π_i and the base σ^{*}, where σ^{*} is a closed hypersurface in event space. Let D be the region of the state space F_i bounded by the hypersurface S^{*}, and δ the region of event space bounded by the hypersurface σ^{*}. First of all, we have two equalities analogous to (2) and (4) (with S replaced by S^{*} and σ by σ^{*}). According to the integral theorem (^{2}), which relates the integral over a closed hypersurface to the integral over the region bounded by this hypersurface, the two indicated equalities may be written in the form

$$\int_{D} \sum_{r=0}^{6} f^{r} \frac{\partial}{\partial x^{r}} A_{i}(x, P)\, dX\, dP = \int_{\delta} \nabla_{\alpha} n^{\alpha}_{i}\, dX, \tag{9}$$

$$\int_{D} \sum_{r=0}^{6} f^{r} \frac{\partial}{\partial x^{r}} \left[(\xi(x), P) A_{i}(x, P)\right] dX\, dP = \int_{\delta} \nabla_{\alpha} \eta^{\alpha}_{i}\, dX, \tag{10}$$

where

$$dX = \sqrt{-g}\, dx^{0}\, dx^{1}\, dx^{2}\, dx^{3}. \tag{11}$$

* See the preceding footnote.

Taking the kinetic equation into account, we transform equality (9) to the form

$$\int_{\delta} dX \sum_{j=1}^{N} \int_{\Pi_{i}} I_{ij}\, dP = \int_{\delta} \nabla_{\alpha} n^{\alpha}_{i}\, dX. \tag{12}$$

Let us consider the left-hand side of equality (10). Since

$$\sum_{r=0}^{6} f^{r} \frac{\partial}{\partial x^{r}} p^{0} = \frac{dp^{0}}{d\tau} = -\Gamma^{0}_{\beta\gamma} p^{\beta} p^{\gamma}, \tag{13}$$

then

$$\sum_{r=0}^{6} f^r \frac{\partial}{\partial x^r}(\xi(x), P) = \frac{1}{2}(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) p^\alpha p^\beta. \tag{14}$$

Taking into account the kinetic equation, as well as formulas (14) and (5), we can write formula (10) in the form

$$\int_\delta dX \sum_{j=1}^{N} \int_{\Pi_i} (\xi(x), P) I_{ij}\, dP = \int_\delta \xi_\beta \nabla_\alpha T_i^{\alpha\beta}\, dX. \tag{15}$$

Let us sum both sides of equality (15) over i from 1 to N. As a result we obtain

$$\int_\delta dX \sum_{i,j=1}^{N} \int_{\Pi_i} (\xi(x), P) I_{ij}\, dP = \int_\delta \xi_\beta \nabla_\alpha T^{\alpha\beta}\, dX. \tag{16}$$

In formulas (12) and (16) we have obtained integrals of the form

$$S_{ij} = \int_{\Pi_i} \psi_i(P) I_{ij}\, dP. \tag{17}$$

In formula (12) the function $\psi_i(P)$ is equal to unity for any P, while in formula (16) it is equal to $(\xi(x), P)$. Using formulas (13) of paper (^{1}), it is not difficult to see that the integral S_{ij} is equal to

$$S_{ij} = \iint_{\Pi_i \Pi_j} A_i(x, P) A_j(x, Q) \langle P, Q \rangle \psi_{ij}(P, Q)\, dP\, dQ, \tag{18}$$

where

$$\psi_{ij}(P, Q) = \int_{\Pi_i} [\psi_i(P') - \psi_i(P)] H_{ij}(\langle P, Q \rangle, \langle P', Q' \rangle, \langle P', P \rangle)\, dP'. \tag{19}$$

Thus, if $\psi_i(P)$ does not depend on P, then $S_{ij} = 0$.

Consequently, the left-hand side of equality (12) is equal to zero. In view of the arbitrariness of the domain δ, hence we find that $\nabla_\alpha n_i^\alpha = 0$. This equality was obtained because we considered stable particles and assumed that the collisions of particles are elastic.

Taking further into account that the function H_{ij} in (19) is proportional to the δ-function of $(P - P', Q + P)$, we find

$$\psi_{ij}(P, Q) = \int_0^\pi \int_0^{2\pi} [\psi_i(P') - \psi_i(P)] h_{ij}(\langle P, Q \rangle \cos\vartheta) \sin\vartheta\, d\vartheta\, d\varphi, \tag{20}$$

where the momentum P' as a function of P, Q, ϑ, and φ is determined by one of the pair of formulas (15) of paper (^{1}). The other formula from this pair determines the momentum Q' in the expression

$$\psi_{ji}(Q,P) = \int_0^{\pi}\int_0^{2\pi}[\psi_j(Q') - \psi_j(Q)]h_{ji}(\langle Q,P\rangle, \cos\vartheta)\sin\vartheta\, d\vartheta\, d\varphi. \tag{21}$$

We note that the function $\langle P, Q\rangle$ is symmetric with respect to the arguments P, Q, and the function h_{ij} is symmetric with respect to the indices i, j. Changing in formula (18) the order of the indices i, j and the order of the integration variables P, Q, we obtain the expression needed for S_{ji}. Adding this expression to expression (18), we find that the sum $S_{ji} + S_{ij}$ is represented by an integral with some weight of the combination $\psi_i(P') + \psi_j(Q') - \psi_i(P) - \psi_j(Q)$. If $\psi_i(P) = (\xi(x), P)$, then this combination is equal to zero, and consequently $S_{ij} + S_{ji} = 0$.

Thus, the left-hand side of equality (16) is equal to zero. In view of the arbitrariness of the vector field $\xi(x)$ and the arbitrariness of the domain δ, it follows that $\nabla_\alpha T^{\alpha\beta} = 0$. Thus, tensor (6) possesses all the properties of the mass tensor of a closed conservative system, if the distribution function satisfies the kinetic equation.

Having the kinetic equation and the expression of the mass tensor of the gas through its distribution function, one can pose the problem of determining the gravitational field created by the gravitating gas. This problem reduces to a system of equations consisting of the kinetic equation (1) and Einstein' s gravitational equation

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = -\frac{8\pi\gamma}{c}T^{\alpha\beta}, \tag{22}$$

where $T^{\alpha\beta}$ is tensor (6) considered here. From (22) it follows that the curvature scalar is equal to

$$R = 8\pi\gamma c\sum_{i=1}^{N} m_i \int_{\Pi_i} A_i(x,P)\, dP. \tag{23}$$

In particular, if all m_i are equal to zero, then $R = 0$.

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Note: Figure translations are in progress. See original paper for figures.